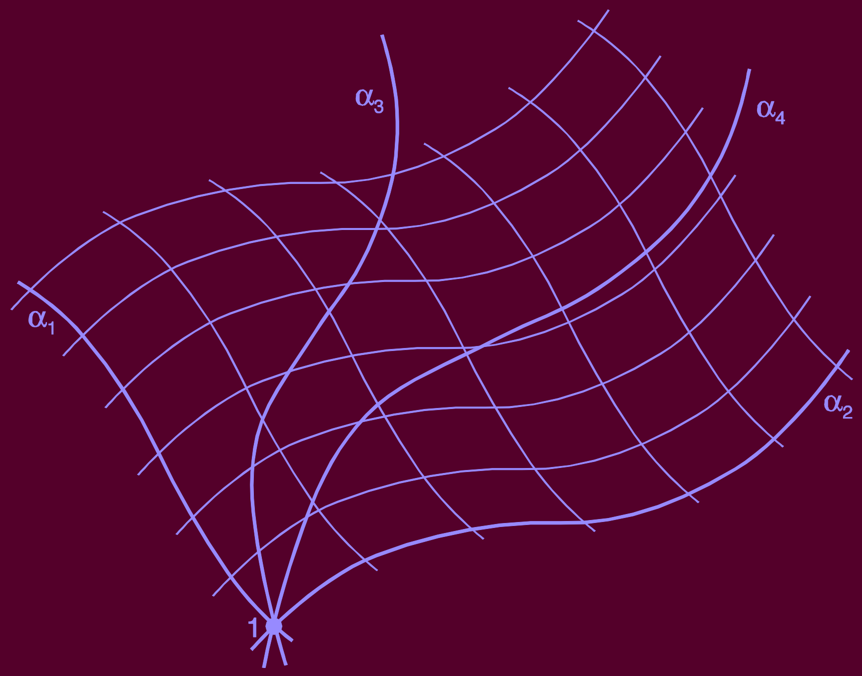
Ranks of Webs and Nets



Eric Moorhouse

University of Wyoming


FISH TRAIL
1.9 MI
MIDDLE RIDGE
FLAT FROG TRAIL
2.3 MI
FROG LAKE
FOREST TRAIL
1.9 MI
ZANITA POINT

Nets

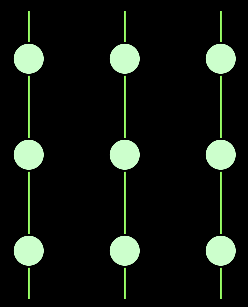
A k-net of order n has

- n^2 points;
- nk lines, each with n points.

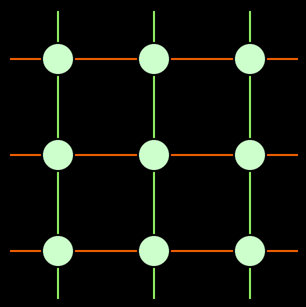
There are k parallel classes of n lines each.

Two lines from different parallel classes meet in a unique point.

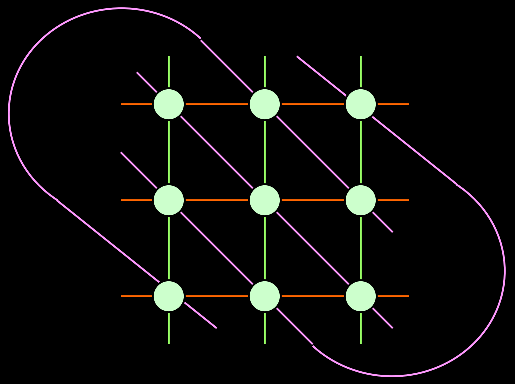
E.g. 1-net of order 3



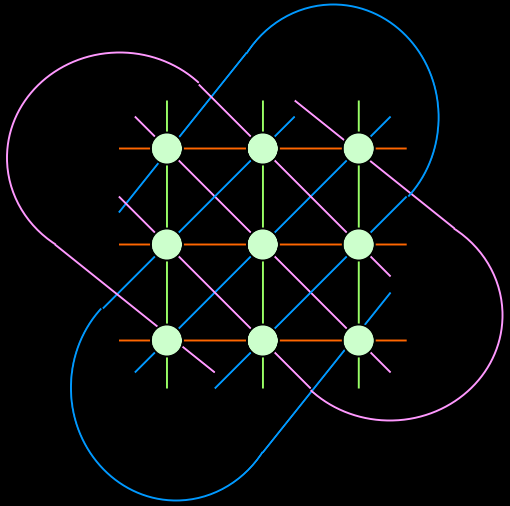
2-net of order 3



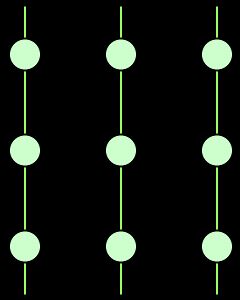
3-net of order 3



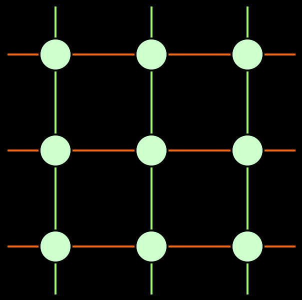
Affine plane of order 3 = 4-net of order 3



E.g. 1-net of order 3



2-net of order 3



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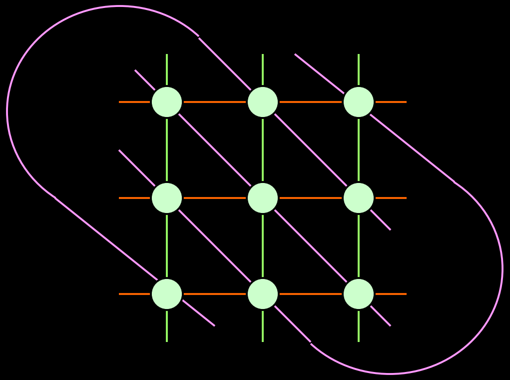

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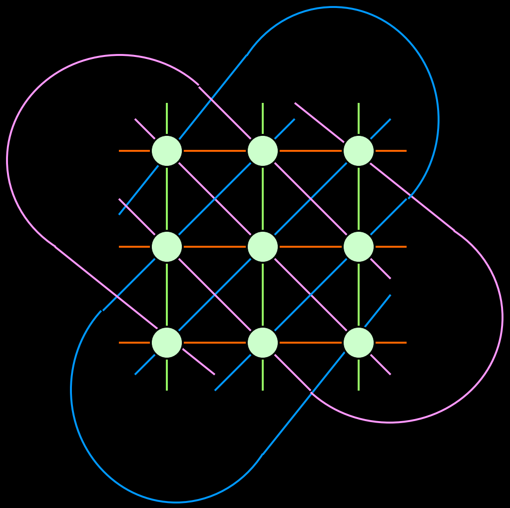

Affine plane of order n = $(n+1)$-net of order n

- n^2 points;
- $n(n+1)$ lines ($n+1$ parallel classes of n lines each).

Any 2 points are joined by exactly one line.
Any two non-parallel lines meet in a unique point.

Open Questions

1. Given an affine (or projective) plane of order n, must n be a prime power?
2. Must every affine (or projective) plane of prime order p be classical?

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Open Questions

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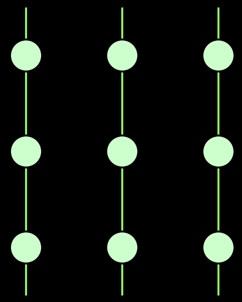
2. Must every affine (or projective) plane of prime order p be classical?

One conceivable approach uses ranks of nets…

rank of a net = rank of its incidence matrix.

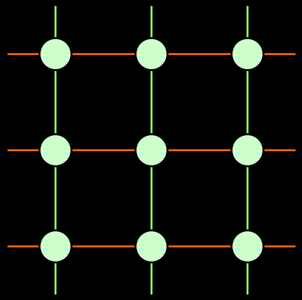
p-rank of a net = rank of its incidence matrix over $\mathbb{F}_p = \{0, 1, 2, \ldots, p\text{-}1\}$

1-net of order 3



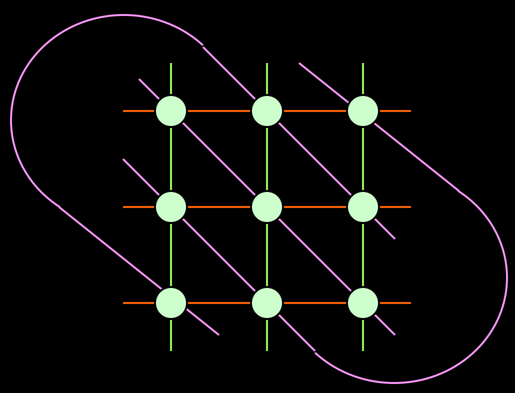
$$\mathrm{rank}_3 \begin{pmatrix} 1\,1\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 1\,1\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 1\,1\,1 \end{pmatrix} = 3$$

2-net of order 3



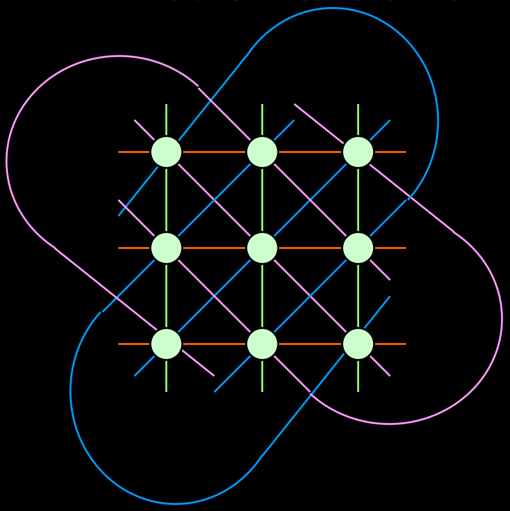
$$\mathrm{rank}_3 \begin{pmatrix} 1\,1\,1 & 0\,0\,0 & 0\,0\,0 \\ 0\,0\,0 & 1\,1\,1 & 0\,0\,0 \\ 0\,0\,0 & 0\,0\,0 & 1\,1\,1 \\ 1\,0\,0 & 1\,0\,0 & 1\,0\,0 \\ 0\,1\,0 & 0\,1\,0 & 0\,1\,0 \\ 0\,0\,1 & 0\,0\,1 & 0\,0\,1 \end{pmatrix} = 3+2 = 5$$

3-net of order 3



$$
\mathrm{rank}_3 \begin{pmatrix}
1&1&1&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&1&1&1\\
1&0&0&1&0&0&1&0&0\\
0&1&0&0&1&0&0&1&0\\
0&0&1&0&0&1&0&0&1\\
1&0&0&0&1&0&0&0&1\\
0&1&0&0&0&1&1&0&0\\
0&0&1&1&0&0&0&1&0
\end{pmatrix} = 3+2+1 = 6
$$

4-net of order 3



$$\mathrm{rank}_3 \begin{pmatrix} 1&1&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 1&0&0&1&0&0&1&0&0\\ 0&1&0&0&1&0&0&1&0\\ 0&0&1&0&0&1&0&0&1\\ 1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&1&0&0\\ 0&0&1&1&0&0&0&1&0\\ 1&0&0&0&1&0&0&0&1\\ 0&1&0&1&0&0&0&1&0\\ 0&0&1&0&0&1&1&0&0 \end{pmatrix} = 3+2+1+0 = 6$$

Conjecture: Any k-net of prime order p has p-rank *at least*

$$p + (p\text{-}1) + (p\text{-}2) + \ldots + (p\text{-}k\text{+}1) = pk - \tfrac{1}{2}k(k\text{-}1)$$

for $k = 1, 2, 3, \ldots, p+1$.

Moreover, nets whose p-rank achieves this lower bound are 'classical'.

I.e. the incidence matrix of any k-net of order p has nullity *at most*

$$\tfrac{1}{2}k(k\text{-}1).$$

Rephrase:

$\mathfrak{P}$ = point set, $|\mathfrak{P}| = p^2$.
Let $i = 1, 2, \ldots, k$ label the parallel class.

Say point $P \in \mathfrak{P}$ lies in the j^{th} line of class i, $j \in \{0, 1, 2, \ldots, p\text{-}1\}$. Define $u_i(P) = j$.

This gives 'coordinate functions' $u_i : \mathfrak{P} \to F = \mathbb{F}_p$.

Any two coordinates $u_i(P), u_j(P)$ (for $i \neq j$) determine $P \in \mathfrak{P}$ uniquely.

The *null space* of the net is the space $\mathcal{V}$ of k-tuples $(f_1, f_2, \ldots, f_k)$ of functions $F \to F$ such that

$$f_1(u_1(P)) + f_2(u_2(P)) + \ldots + f_k(u_k(P)) = 0$$

for every point $P \in \mathfrak{P}$. Conjecture: $\dim \mathcal{V} \leq \frac{1}{2}k(k\text{-}1)$.

We can replace any such k-tuple $(f_1, f_2, \ldots, f_k)$ by $(f_1+c_1, f_2+c_2, \ldots, f_k+c_k)$ where the constants

$$c_1 + c_2 + \ldots + c_k = 0.$$

So replace $\mathcal{V}$ by the subspace $\mathcal{V}_0$ consisting of all k-tuples $(f_1, f_2, \ldots, f_k)$ of functions $F \to F$ such that

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for every point $P \in \mathfrak{P}$, *and* $f_i(0)=0$.

Conjecture: dim $\mathcal{V}_0 \leq \frac{1}{2}(k-1)(k-2)$.

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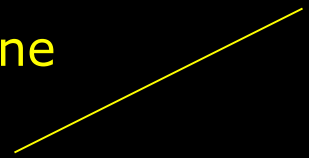
Conjecture: $\dim \mathcal{V}_0 \leq \frac{1}{2}(k-1)(k-2)$.

Note: An algebraic plane curve of degree k has genus $\leq \frac{1}{2}(k-1)(k-2)$.

This is not mere coincidence…

k = degree of curve	real curve	genus $g \leq \frac{1}{2}(k\text{-}1)(k\text{-}2)$	Riemann surface
1	line 	0	

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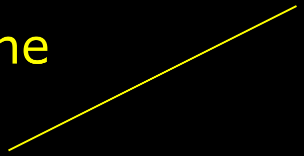
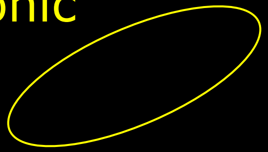
1	line 	0	
2	conic 	0	

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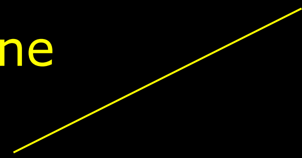
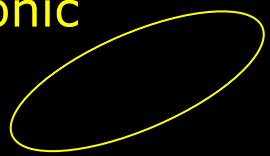
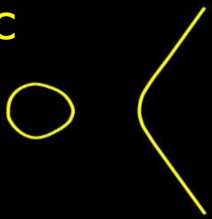
1	line 	0	
2	conic 	0	
3	elliptic curve 	1	

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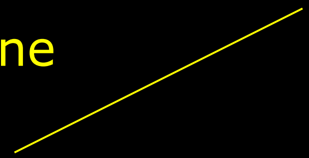
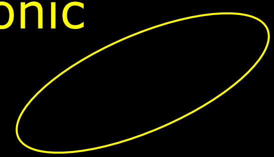
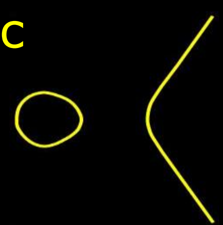
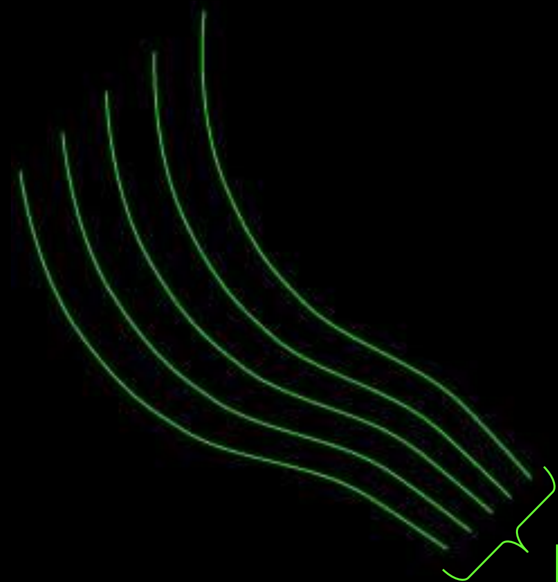
1	line 	0	
2	conic 	0	
3	elliptic curve 	1	
3	singular cubic 	0	

Take $F = \mathbb{R}$ or $\mathbb{C}$.

Consider functions u_i: $F^2 \to F$, $i = 1, 2, \ldots, k$.



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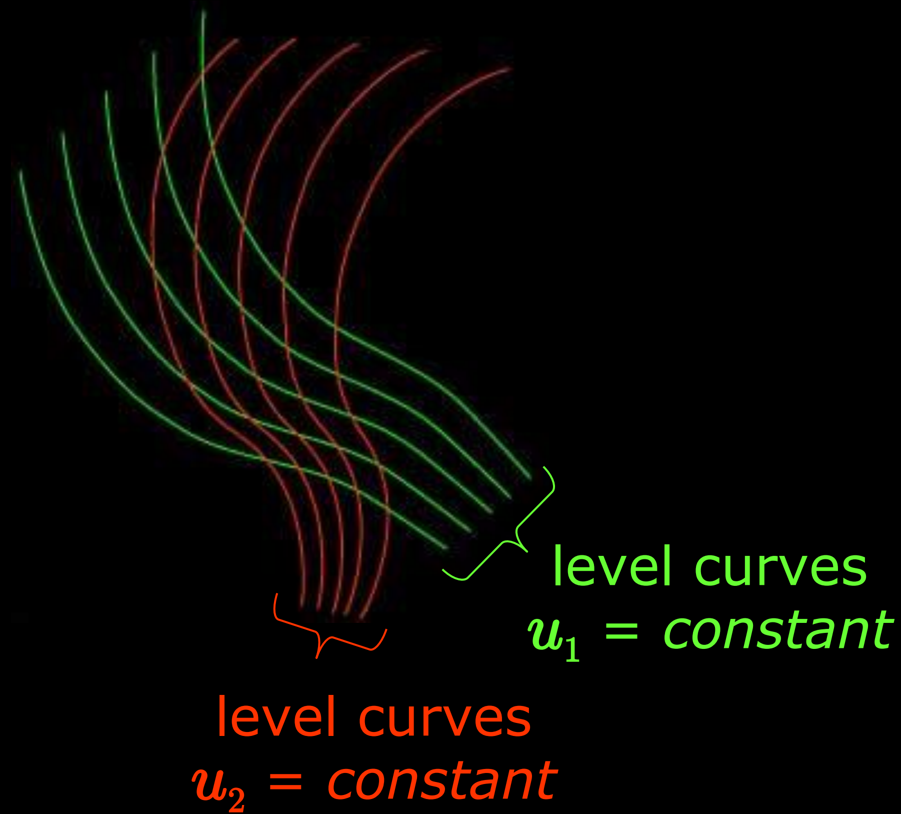

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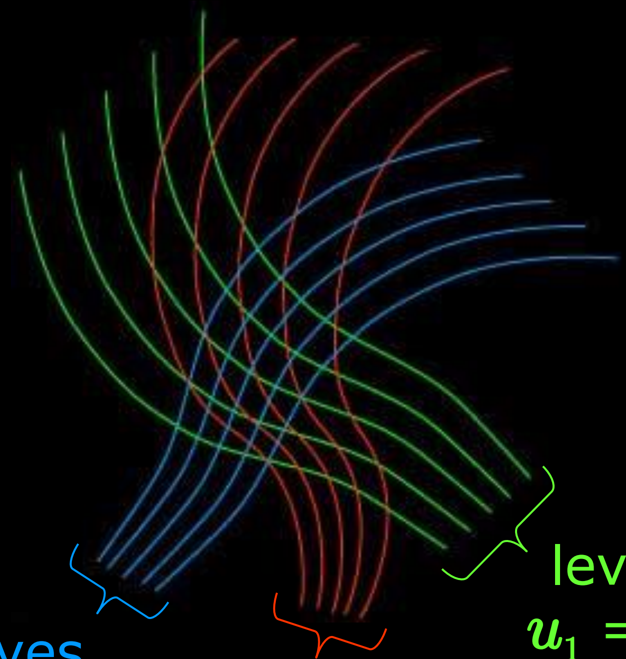

This is a k-*web*

(of codimension 1).

level curves

u_1 = *constant*

level curves

u_3 = *constant*

level curves

u_2 = *constant*

Assume level curves meet transversely, i.e.

∇u_i , ∇u_j are linearly independent for $i \neq j$.

V.V. Goldberg
1936–

approx. 100 publications
on *web geometry* by
V.V. Goldberg & M.A. Akivis

$F = \mathbb{R}$ or $\mathbb{C}$.

coordinate functions $u_i : F^2 \to F$, i=1,2,…,k.

$\mathcal{V}_0$ = vector space of all k-tuples (f_1, f_2, …, f_k) of smooth functions $F \to F$ such that

$$f_1(u_1(P)) + f_2(u_2(P)) + \ldots + f_k(u_k(P)) = 0$$

for every point $P \in F^2$, *and* $f_i(0)=0$.

Theorem (Blaschke et al.) dim $\mathcal{V}_0 \leq \frac{1}{2}(k-1)(k-2)$.

If equality holds then the k-web is 'algebraic'; it arises from an algebraic curve of maximal genus.

Note: dim $\mathcal{V}_0$ is called the *rank* of the k-web.

W. Blaschke
1885–1962

W. Blaschke & G. Bol, *Geometrie der Gewebe,* 1938

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N. Abel
1802–1829

Abel's Theorem is the foundation for the Theorem of Blaschke et al.

Chern & Griffiths:
Numerous publications on
Abel's Theorem and webs

P. Griffiths
1938–

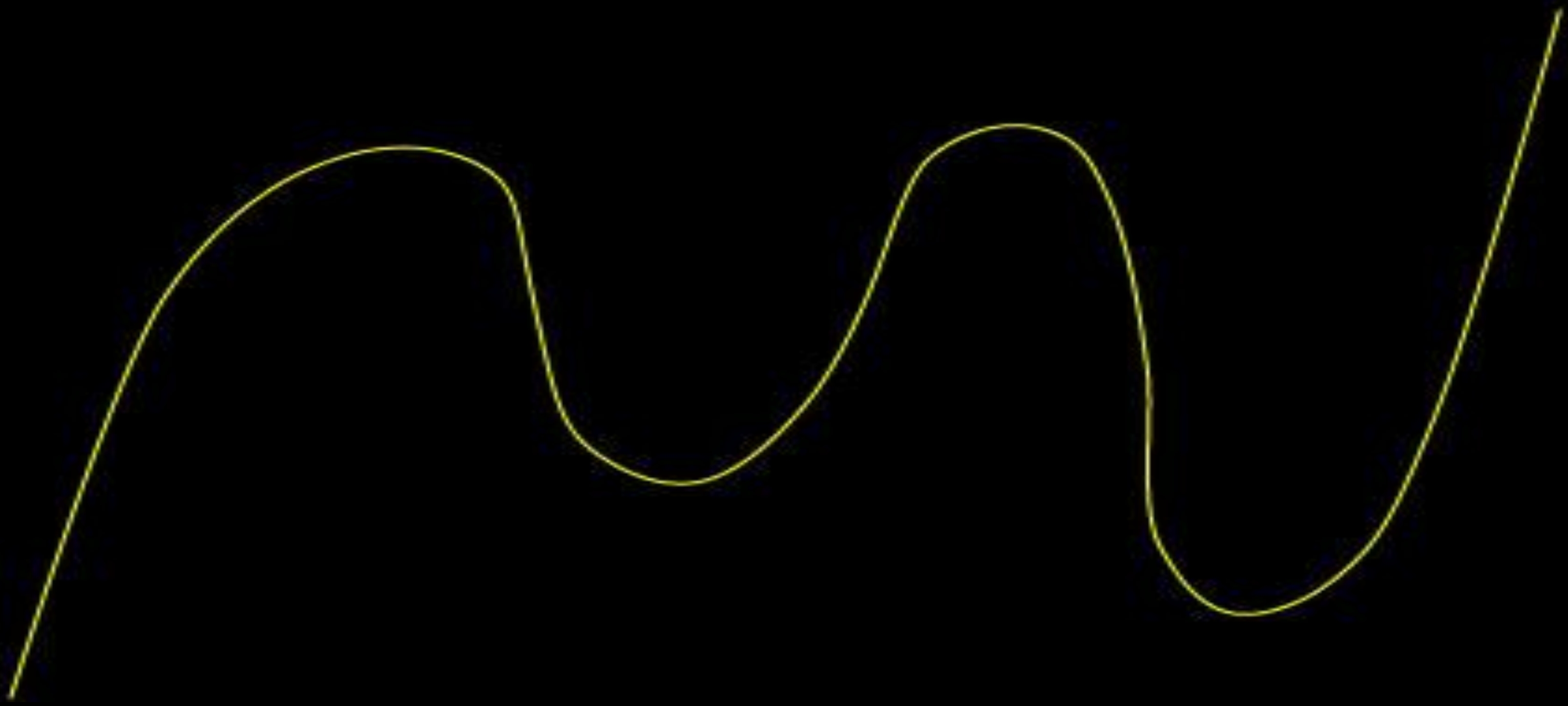
S.S. Chern
1911–

J. Little
1956–

Little’s dissertation, under B. Saint-Donat, and several subsequent papers, concern webs of maximal rank.

In particular he proved an analogue (1984) over algebraically closed fields of positive characteristic.

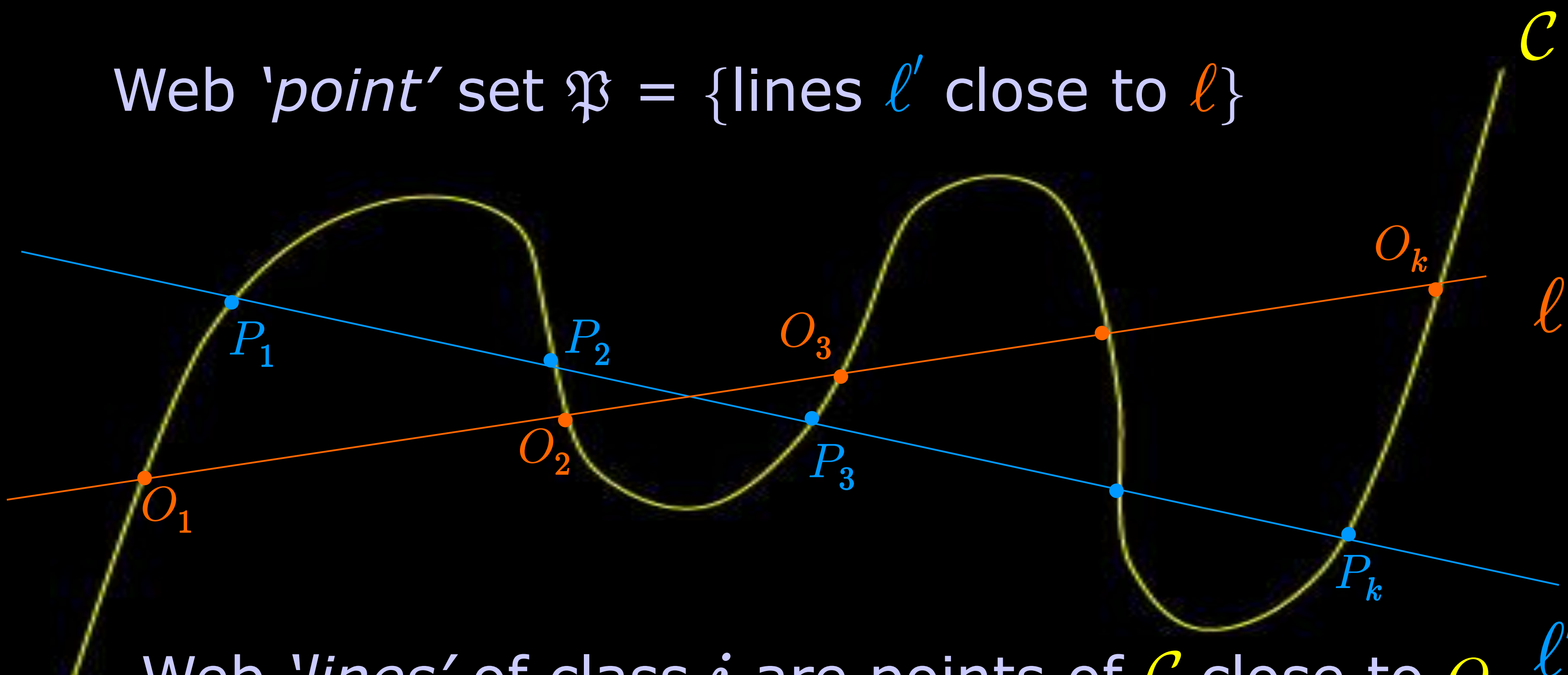
How does an algebraic curve give rise to a web?



Consider an algebraic curve $\mathcal{C}$ of degree k in the plane.

A typical line ℓ meets $\mathcal{C}$ in k points.

Web *'point'* set $\mathfrak{P}$ = {lines ℓ' close to ℓ}



Web *'lines'* of class i are points of $\mathcal{C}$ close to O_i .

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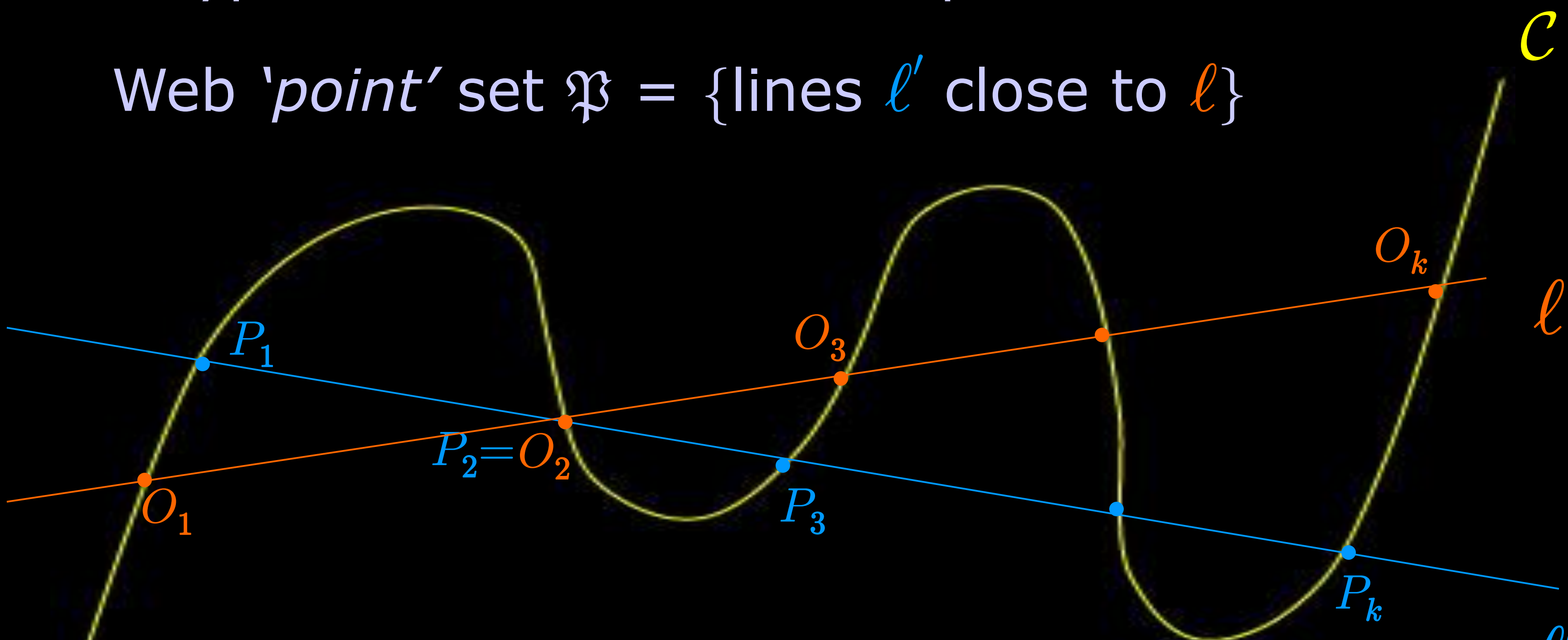

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If $\ell' \cap \mathcal{C} = \ell \cap \mathcal{C}$, interpret ℓ' and ℓ as 'points' joined by a 'line' of class i.

Consider an algebraic curve $\mathcal{C}$ of degree k in the plane.

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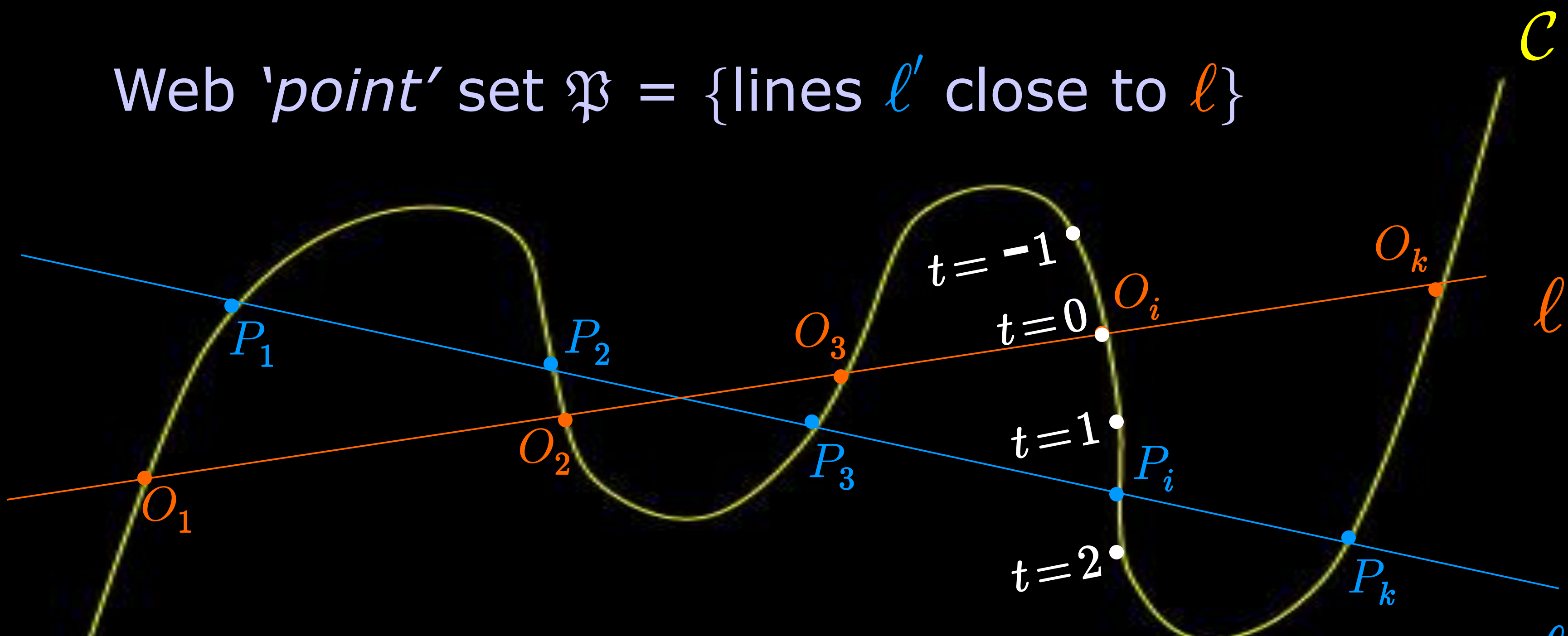


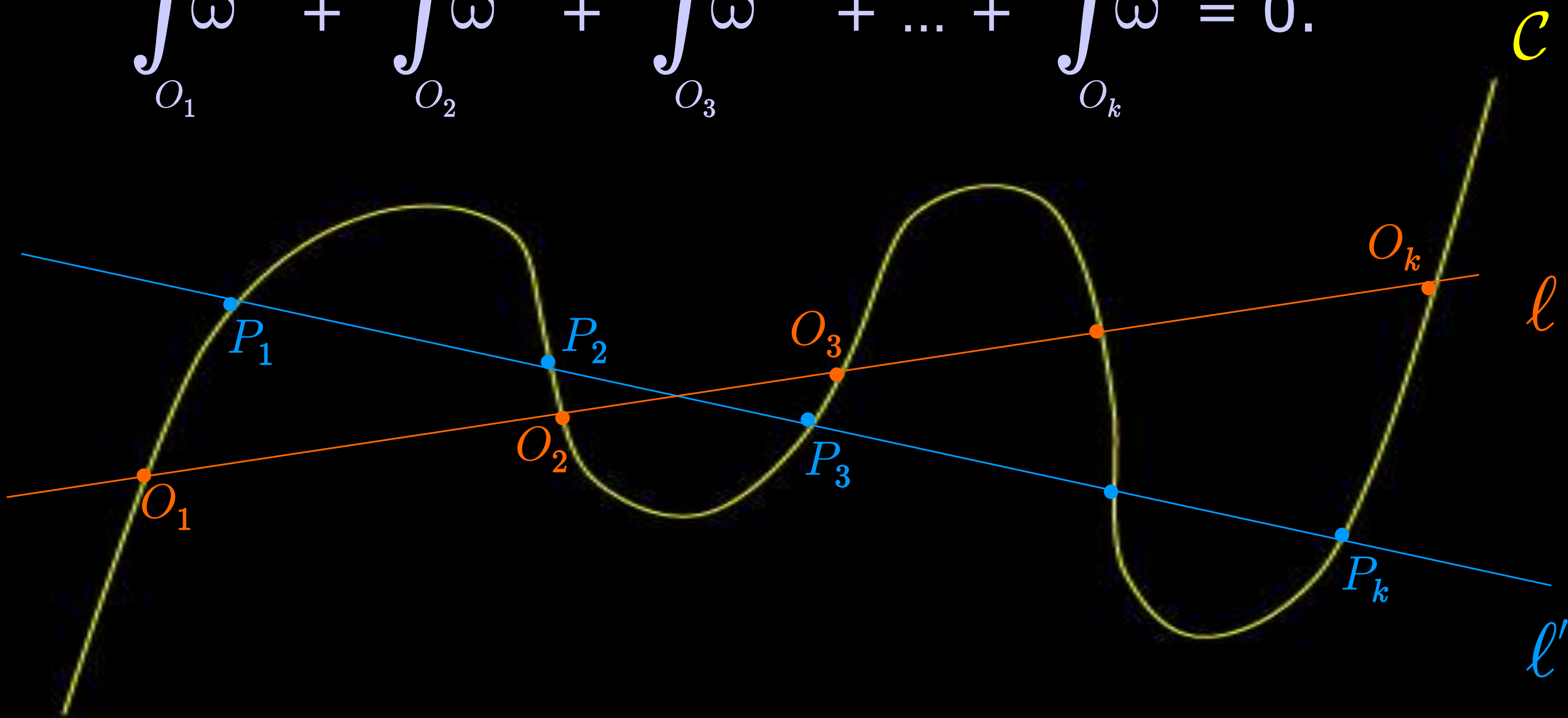
Web *'lines'* of class i are points of $\mathcal{C}$ close to O_i.

Parameterise points of $\mathcal{C}$ near O_i.

Define $u_i(\ell')$ = parameter value of $P_i = \ell' \cap \mathcal{C}$

Abel's Theorem. If ω is any holomorphic 1-form on $\mathcal{C}$, then

$$\int_{O_1}^{P_1}\omega + \int_{O_2}^{P_2}\omega + \int_{O_3}^{P_3}\omega + \ldots + \int_{O_k}^{P_k}\omega = 0.$$



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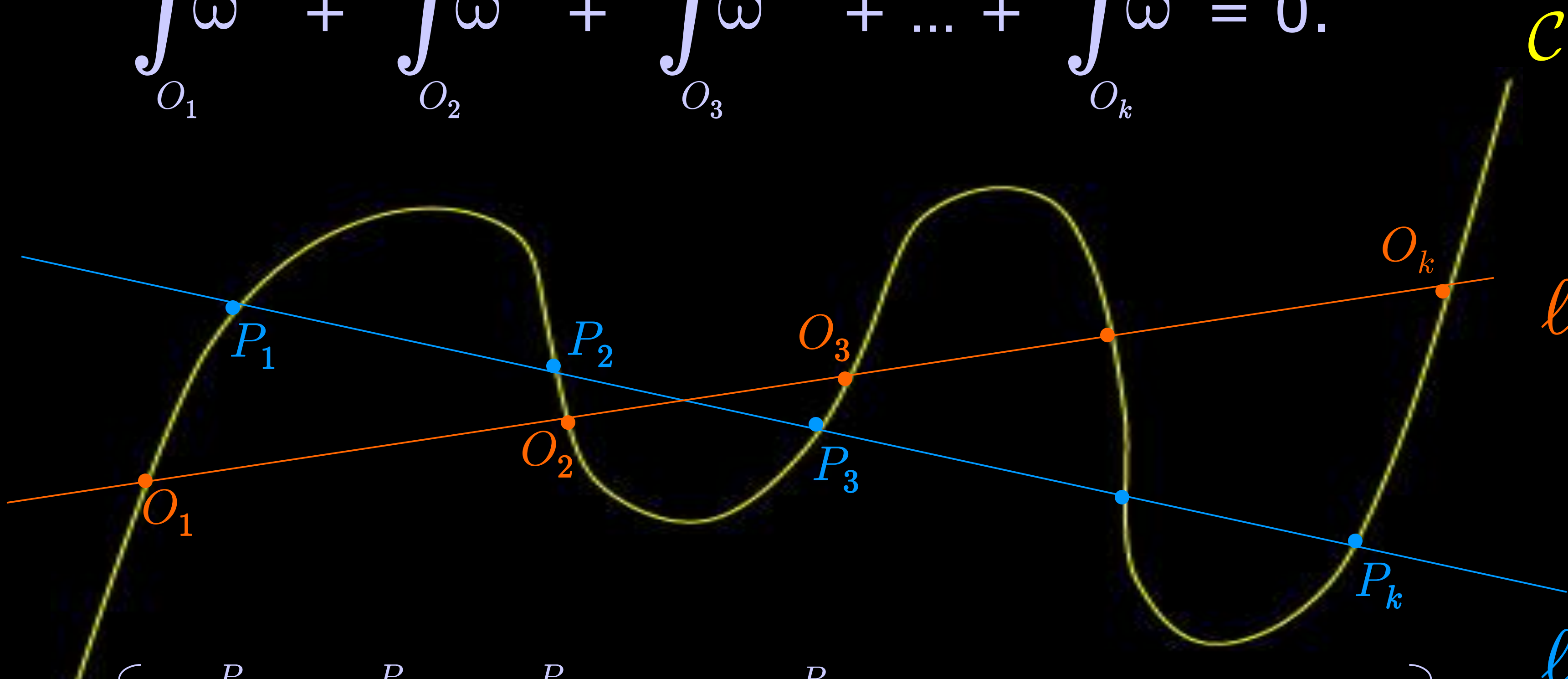


$$\mathcal{V}_0 = \left\{ \left(\int_{O_1}^{P_1} \omega , \int_{O_2}^{P_2} \omega , \int_{O_3}^{P_3} \omega , \dots, \int_{O_k}^{P_k} \omega \right) : \omega \text{ any 1-form} \right\}$$

$$\dim \mathcal{V}_0 = \text{genus of } \mathcal{C} \leq \tfrac{1}{2}(k-1)(k-2)$$

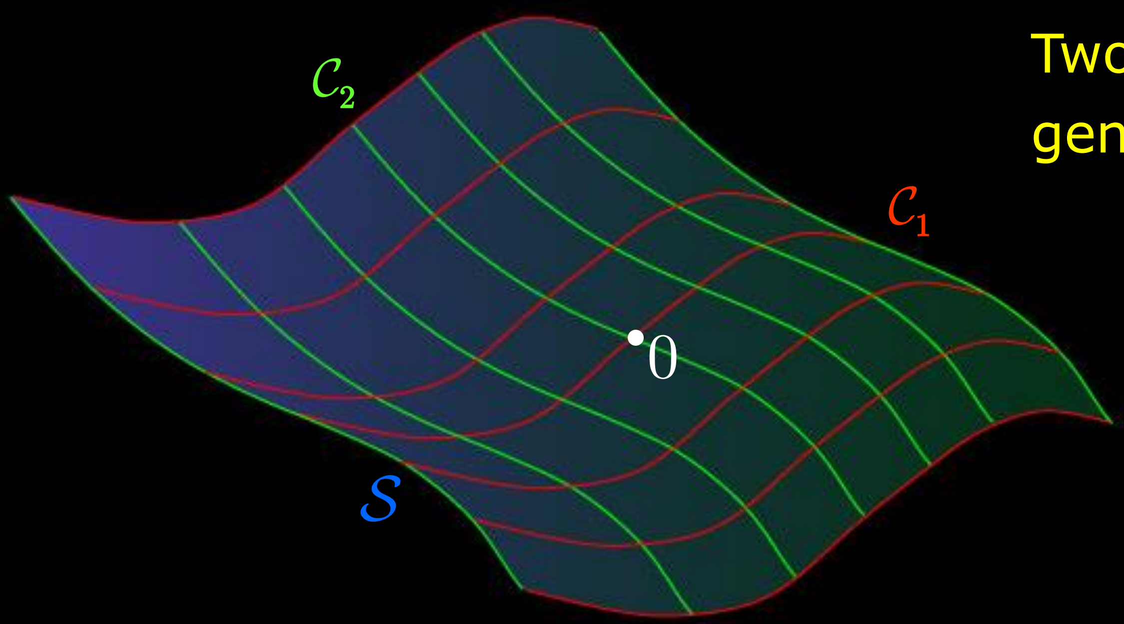
Special case $\boldsymbol{k} = 4$

A 4-*web* of maximal rank

or

a 4-*net* of order $\boldsymbol{p}$, and $\boldsymbol{p}$-rank attaining the conjectured lower bound

yields:



Two curves $\mathcal{C}_1$, $\mathcal{C}_2$ in 3-space generate surface

$$\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$$

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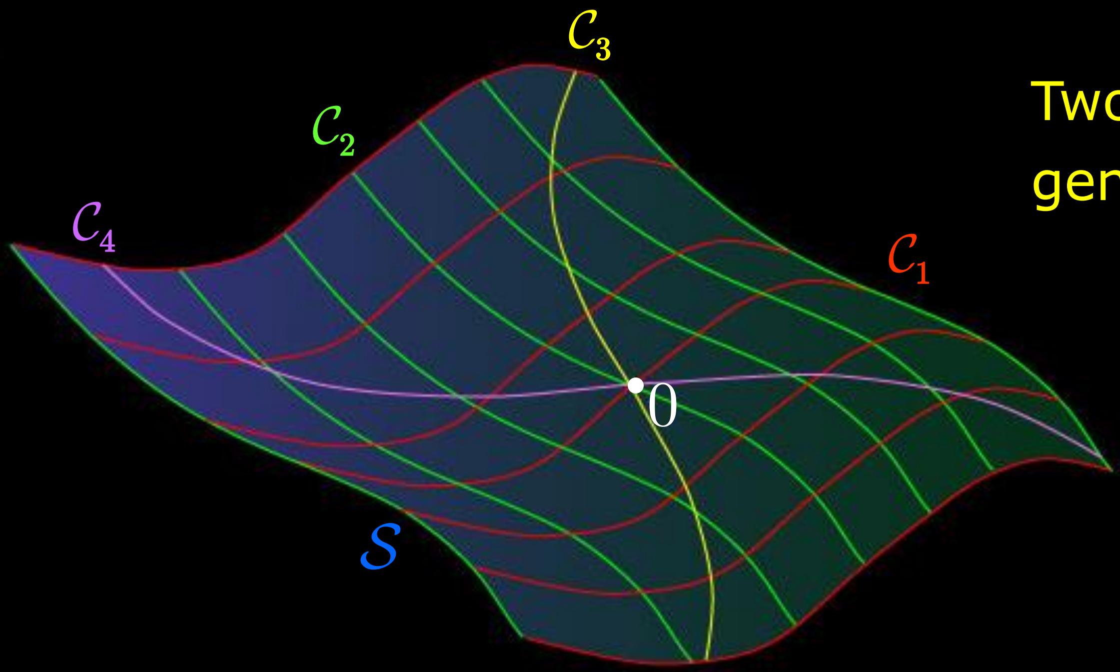

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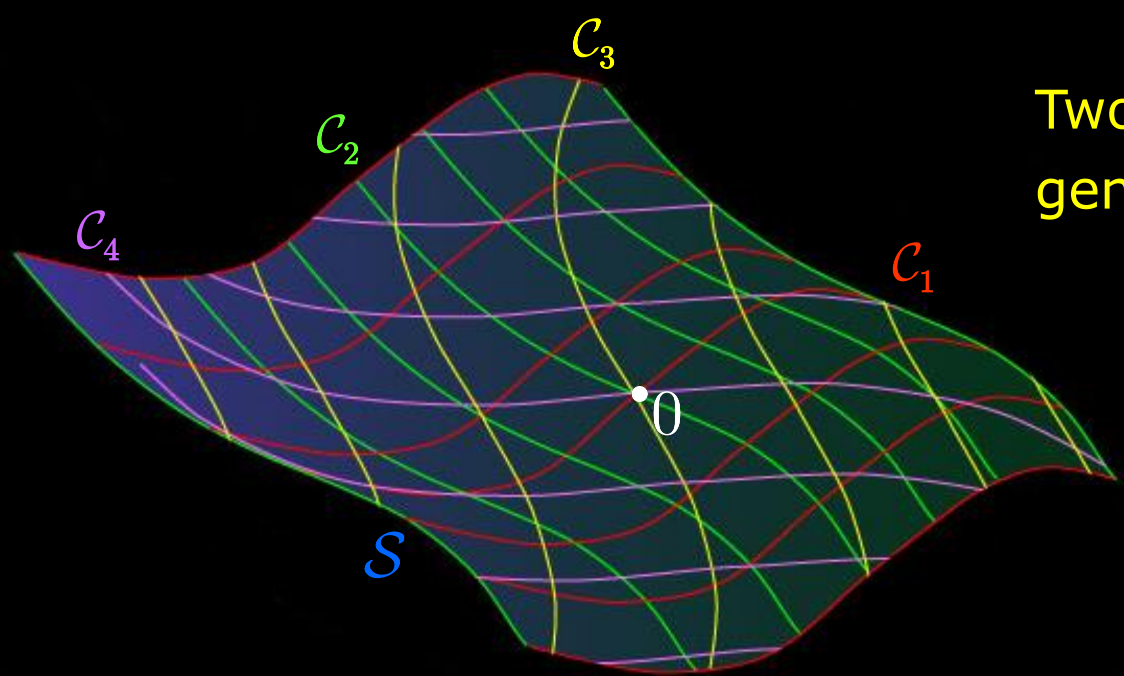

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$$\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$$

$$= \mathcal{C}_3 + \mathcal{C}_4$$

Example

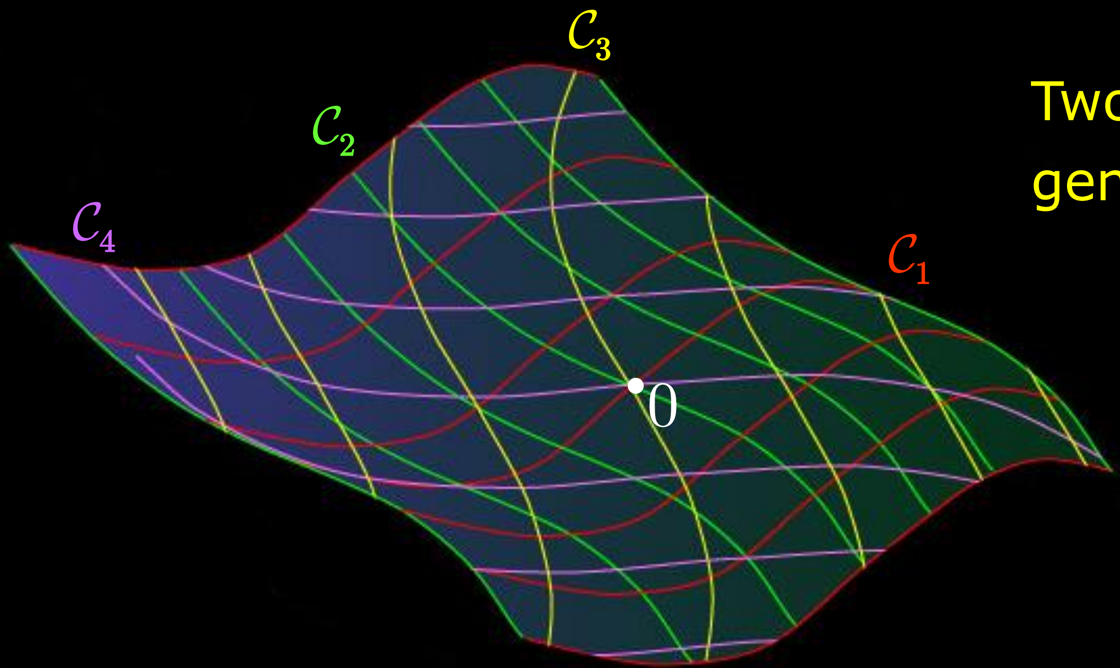
$\mathcal{S}$: $z = cx^2 - y^2$

$\mathcal{C}_1 = \{(x, 0, cx^2) : x \in F\}$

$\mathcal{C}_2 = \{(0, y, -y^2) : y \in F\}$

$\mathcal{C}_3 = \{(s, cs, c(1-c)s^2) : s \in F\}$

$\mathcal{C}_4 = \{(t, t, (c-1)t^2) : t \in F\}$



Two curves $\mathcal{C}_1$, $\mathcal{C}_2$ in 3-space generate surface

$$\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$$

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Example 2

$\mathcal{S}$:

$$2z = y^4 + 2xy^2 - x^2$$

$$\mathcal{C}_1 = \{(s^2, s, s^4) : s \in \mathbb{R}\}$$

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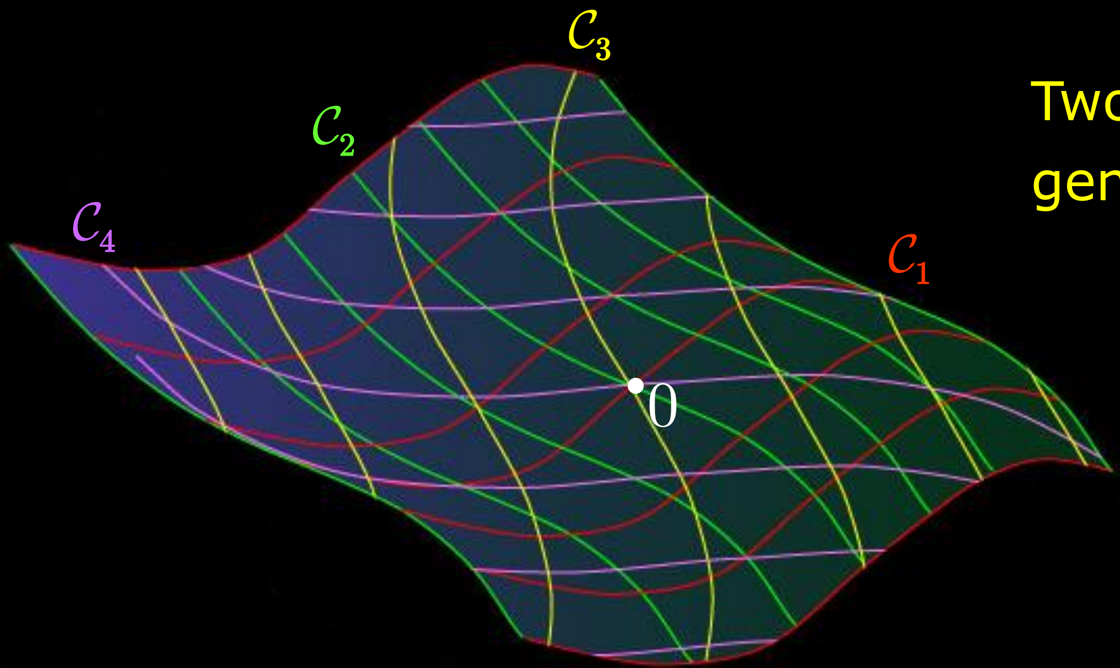

Two curves $\mathcal{C}_1$, $\mathcal{C}_2$ in 3-space generate surface

$$\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$$
$$= \mathcal{C}_3 + \mathcal{C}_4$$

Example 3

$\mathcal{S}$:

$$z = (x+1)e^{-cy} - 1 + \tfrac{c}{2}x^2 + cx$$

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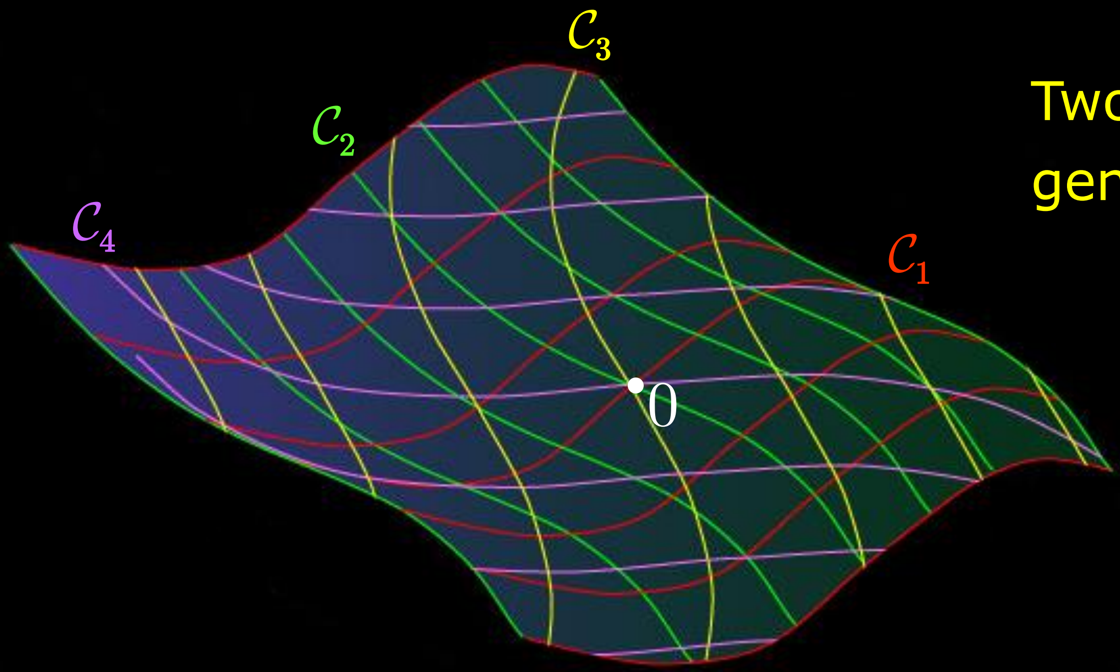
$$\mathcal{C}_2 = \{(\tfrac{1}{c}(1-e^{-cs}), s, \tfrac{1}{2c}(1-e^{-2cs})) : s \in \mathbb{R}\}$$

$$\mathcal{C}_3 = \{(0, y, e^{-cy}-1) : y \in \mathbb{R}\}$$

$$\mathcal{C}_4 = \{(t, \tfrac{1}{c}\ln(1+t), \tfrac{c}{2}t^2 + ct) : t > -1\}$$



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$$\mathcal{S} = \mathcal{C}_1 + \mathcal{C}_2$$
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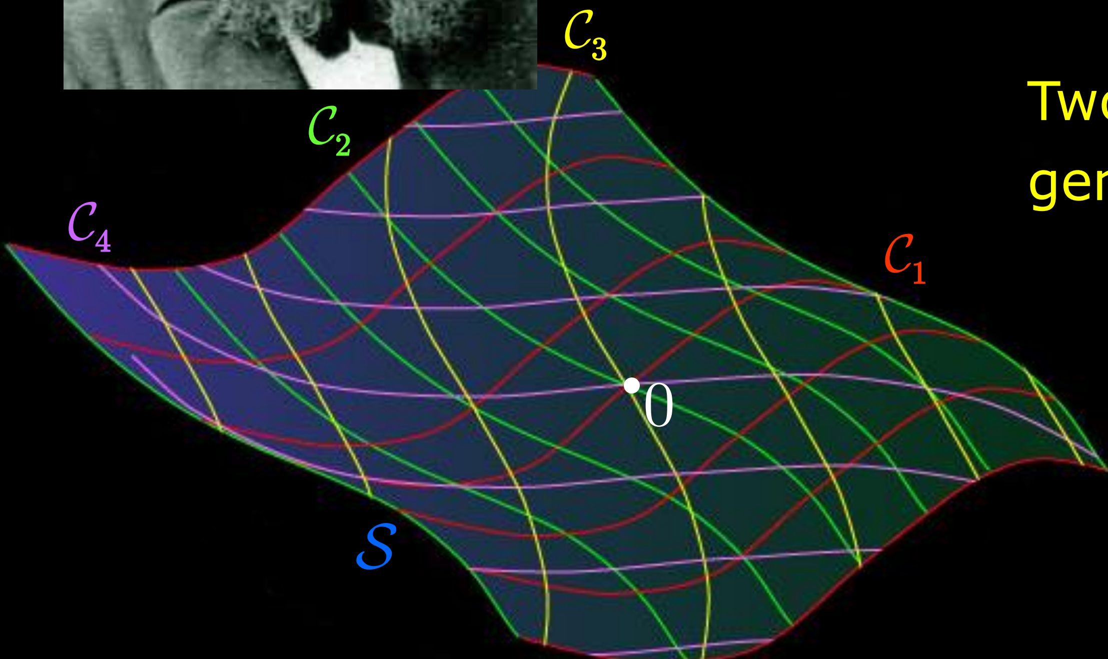
S. Lie
1842–1899

Lie (1882) first considered such a *double translation surface.*



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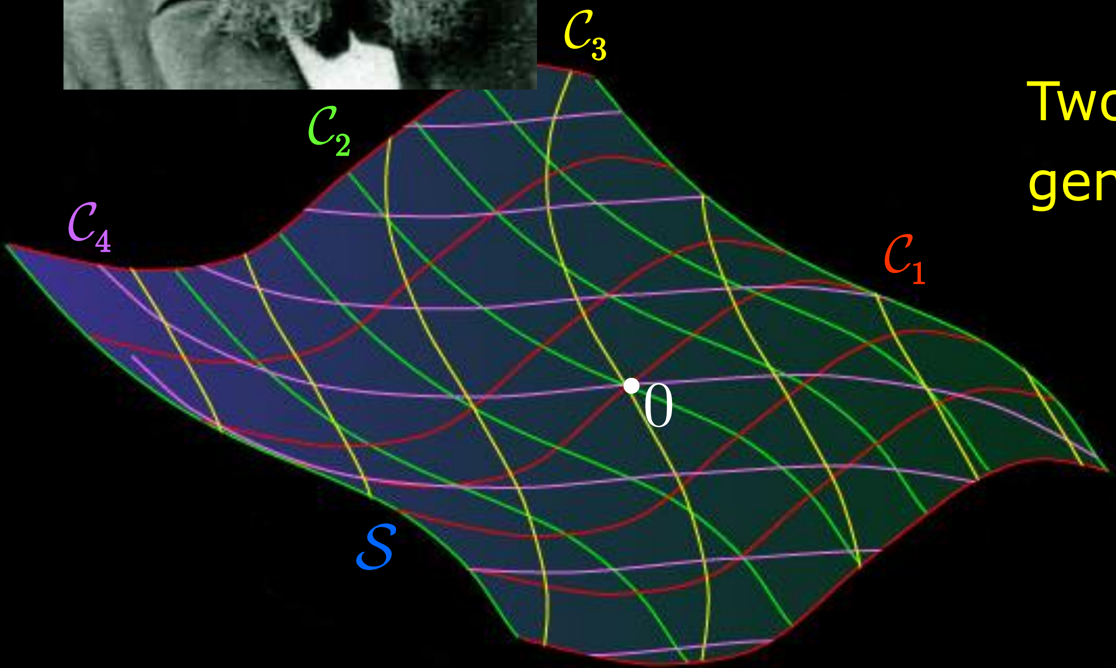
S. Lie
1842–1899

Theorem (Lie, 1882). Consider any double translation surface in $\mathbb{C}^3$, as below. There is an algebraic curve $\mathcal{C}$ of degree 4 in the plane at infinity, such that all tangent lines to $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ all pass through $\mathcal{C}$.



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S. Lie
1842–1899

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Conversely, *every* algebraic curve $\mathcal{C}$ of degree 4 and algebraic genus 3 in the plane at infinity determines a double translation surface $\mathcal{S}$ in this way.

Chern called this result a *'true tour de force'.*

S. Lie
1842–1899

H. Poincaré
1854–1912

Lie was not thrilled.

Poincaré published sequels (1895, 1901) to Lie's paper, observing the connection to Abel's Theorem.

Example 1

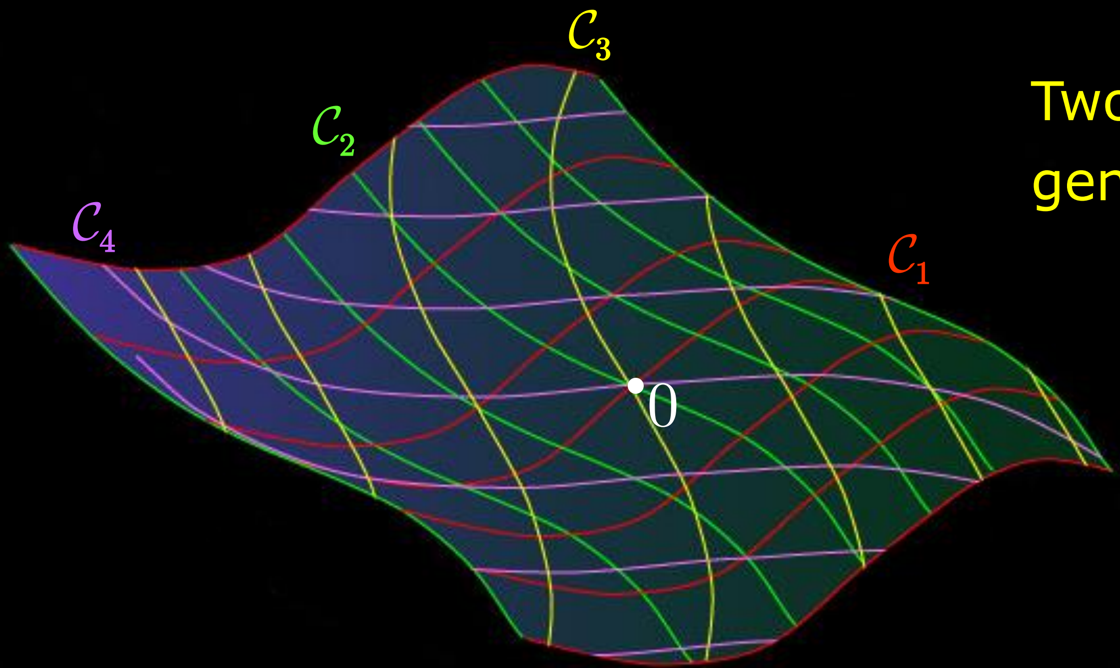
$\mathcal{S}$: $z = cx^2 - y^2$

$\mathcal{C}_1 = \{(x, 0, cx^2) : x \in F\}$

$\mathcal{C}_2 = \{(0, y, -y^2) : y \in F\}$

$\mathcal{C}_3 = \{(s, cs, c(1-c)s^2) : s \in F\}$

$\mathcal{C}_4 = \{(t, t, (c-1)t^2) : t \in F\}$



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Tangent vectors $(1, 0, 2cx)$

$(0, 1, -2y)$

$(1, c, 2c(1-c)s)$

$(1, 1, 2(c-1)t)$

all lie in the curve $XY(Y-X)(Y-cX) = 0$ of degree 4

Example 2

$\mathcal{S}$:

$$2z = y^4 + 2xy^2 - x^2$$

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$$\mathcal{C}_4 = \{(-v^2, v, -v^4) : v \in \mathbb{R}\}$$

Tangent vectors

$$(2s, 1, 4s^3)$$

$$(-2, 0, -4t)$$

$$(-2u, 1, -4u^3)$$

$$(-2v, 1, -4v^3)$$

all lie in the curve $Y(X^3 - 2Y^2Z) = 0$ of degree 4

Example 3

$\mathcal{S}$:

$$z = (x+1)e^{-cy} - 1 + \frac{c}{2}x^2 + cx$$

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Tangent vectors

$$(1, 0, cx+c+1)$$
$$(e^{-cs}, 1, e^{-2cs})$$
$$(0, 1, -ce^{-cy})$$
$$(1, \tfrac{1}{c(t+1)}, c(t+1))$$

all lie in the curve $XY(X^2-YZ)=0$ of degree 4

Conjecture: For the web to be globally defined over F, $\mathcal{C}$ must be a union of four lines.

For $F = \mathbb{F}_p$, this is equivalent to:

If $\mathcal{N}_3 \subset \mathcal{N}_4$ are 3- and 4-nets of order p resp., then $\mathrm{rank}_p \mathcal{N}_4 - \mathrm{rank}_p \mathcal{N}_3 \geq p-1$.

Tangent vectors

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$$(0, 1, -2y)$$
$$(1, c, 2c(1-c)s)$$
$$(1, 1, 2(c-1)t)$$

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