Two-Graphs and Skew Two-Graphs

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Two-graphs $\longleftrightarrow$ Switching-equivalence classes of ordinary graphs

Skew Two-graphs $\longleftrightarrow$ Switching-equivalence classes of tournaments

Problem: Given translation planes $\pi_1, \pi_2, \cdots, \pi_N$ of order n, determine the isomorphism classes

$n = 16$	Dempwolff, Reifart (1983)
$n = 25$	Oakden, Czerwinski (1992)
$n = 49$	Mathon, Royle; Charnes, Dempwolff (1992)
$n = 27$	Dempwolff (1994)

J. H. Conway's invariant (n odd):

plane $\pi \longrightarrow$ fingerprint $f(\pi)$
(a sequence of integers)
computable in $O(n^3)$ operations

$$\pi \rightarrow \left\{ \begin{array}{c} \text{Two-graph } \Delta(\pi), \\ n \not\equiv 3 \bmod 4 \\ \text{(trivial if } q \text{ even)} \\ \text{Skew two-graph } \nabla(\pi), \\ n \equiv 3 \bmod 4 \end{array} \right\} \rightarrow \begin{array}{c} \text{degree} \\ \text{sequence} \end{array} \rightarrow \begin{array}{c} \text{fingerprint} \\ f(\pi) \end{array}$$

In all <u>known</u> cases, $f(\pi') = f(\pi) \Leftrightarrow \pi', \pi$ isomorphic or polar
for $n \equiv 1 \bmod 4$

This fails badly for $n \equiv 3 \bmod 4$.
Eg $n = 27$, $\exists\ \pi_1, \pi_2, \cdots, \pi_7$, $f(\pi_1) = f(\pi_2) = f(\pi_3)$.
Dempwolff introduced Kennvector (computable in $O(n^4)$ time) to distinguish them.

$|X| = v \geq 3$

A <u>two-graph</u> is a $\Delta \subseteq \binom{X}{3}$ such that every 4-subset of X contains an even number (i.e. 0, 2 or 4) of triples from Δ.

Eg. the <u>trivial</u> two-graphs $\begin{cases} \emptyset & \text{(empty)} \\ \binom{X}{3} & \text{(complete)} \end{cases}$

Δ two-graph $\Rightarrow$ $\bar{\Delta} := \binom{X}{3} \setminus \Delta$ <u>complementary</u> two-graph

The <u>degree</u> of $\{x,y\} \subset X$ is the number of triples $\{x,y,z\} \in \Delta$ containing $\{x,y\}$.

Δ is <u>regular</u> $\Leftrightarrow$ every 2-subset $\{x,y\} \subset X$ has the same degree

$\Leftrightarrow$ Δ is a 2-$(v,3,\lambda)$ design, some λ

Γ ordinary graph with vertex set X.

For $X_1 \subseteq X$, $\Gamma(X_1)$ is the graph formed by replacing $\begin{Bmatrix} \text{edges} \\ \text{nonedges} \end{Bmatrix}$ between X_1 and $X \setminus X_1$ by $\begin{Bmatrix} \text{nonedges} \\ \text{edges} \end{Bmatrix}$.

Γ, Γ' are <u>switching-equivalent</u> $\Leftrightarrow$ $\Gamma' = \Gamma(X_1)$, some $X_1 \subseteq X$.

$\Leftrightarrow$ $A' = DAD$, some ± 1-diagonal matrix D

where A, A' are the $(0,\pm 1)$-adjacency matrices of Γ, Γ' (-1 for adj., $+1$ for nonadj., 0 on diag.)

$\Delta(\Gamma) := \{ \{x,y,z\} \in \binom{X}{3} :$ Γ contains an odd number (i.e. 1 or 3) of $\{x,y\}, \{y,z\}, \{z,x\}$

$\Delta(\Gamma) = \Delta(\Gamma')$ $\Leftrightarrow$ Γ, Γ' are switching equivalent

$\Delta(\bar{\Gamma}) = \bar{\Delta}(\Gamma)$

$|X| = v \geq 3$

$\mathrm{Sym}(X) = \{\text{permutations of } X\}$

$\mathcal{T}(X) = \{\text{3-cycles } (x\ y\ z) \in \mathrm{Sym}(X)\}$,

$$|\mathcal{T}(X)| = \frac{(v-1)(v-2)}{2}$$

A <u>skew two-graph</u> (Cameron (1977): 'oriented two-graph') is a subset $\nabla \subset \mathcal{T}(X)$ such that

(i) $\forall \tau \in \mathcal{T}(X)$, exactly one of τ, τ^{-1} is in ∇;

(ii) $\forall \{x,y,z,w\} \in \binom{X}{4}$, ∇ contains an even number (i.e. 0, 2 or 4) of the 3-cycles $(x\ y\ z)$, $(x\ w\ y)$, $(x\ z\ w)$, $(y\ w\ z)$.

(The latter is a conjugacy class of $\mathrm{Alt}(\{x,y,z,w\})$.)

$\bar{\nabla} := \{\tau^{-1} : \tau \in \nabla\}$ is the <u>complementary</u> skew two-graph.

$\nexists$ trivial skew two-graph.

The <u>degree</u> of an ordered pair (x,y) in X, is the number of $z \in X$ such that $(x\ y\ z) \in \nabla$.

∇ is <u>regular</u> $\Leftrightarrow$ every pair (x,y) has the same degree, necessarily $\frac{v-2}{2}$

A <u>tournament</u> T on X is an orientation of the complete graph on X.

For $X_1 \subseteq X$, a tournament $T(X_1)$ is obtained by reversing all edges between X_1 and $X - X_1$.

T, T' <u>switching-equivalent</u> $\Leftrightarrow$ $T' = T(X_1)$, some $X_1 \subseteq X$

$\Leftrightarrow$ $A' = DAD$, some ± 1-diagonal matrix D

where A, A' are the $(0, \pm 1)$-adjacency matrices of T, T' (skew-symmetric).

$\nabla(T) := \{(x\ y\ z) \in \mathcal{T}(X) : T$ contains an odd no. (i.e. 1 or 3) of $(x,y), (y,z), (z,x)\}$

$\nabla(T) = \nabla(T') \Leftrightarrow T, T'$ switching equivalent

$\nabla(\bar{T}) = \bar{\nabla}(T)$

$\nabla(T)$ regular $\Leftrightarrow$ A skew-symmetric conference matrix

$\Leftrightarrow$ $A + I$ (skew) Hadamard matrix

$\Rightarrow$ $v \equiv 0 \bmod 4$

$\mathcal{P}$: classical finite polar space embedded in $PG(V)$.

$f(\cdot,\cdot)$: associated bilinear / sesquilinear form.

Elements of $\mathcal{P}$ are totally isotropic (or tot. singular) subspaces of V:

points, lines, ..., m-flats, ..., generators.

A cap $\mathcal{O}$ in $\mathcal{P}$ is a set of points, no two collinear in $\mathcal{P}$ (perp. with respect to f).

$\mathcal{O}$ is an ovoid if each generator meets $\mathcal{O}$ in a unique point.

Theorem: If $\mathcal{P}$ is of $\left\{\begin{array}{l}\text{orthogonal}\\ \text{unitary}\\ \text{symplectic, } q\not\equiv 3 \bmod 4\end{array}\right\}$ type, then

$$\Delta(\mathcal{O}) := \left\{ \{\langle u\rangle, \langle v\rangle, \langle w\rangle\} \in \binom{\mathcal{O}}{3} : f(u,v)f(v,w)f(w,u) = \not\square \right\}$$

is a two-graph (trivial if q even)

If $\mathcal{P}$ is of symplectic type, $q\equiv 3 \bmod 4$, then

$$\nabla(\mathcal{O}) := \left\{ (\langle u\rangle \langle v\rangle \langle w\rangle) \in \mathfrak{J}(\mathcal{O}) : f(u,v)f(v,w)f(w,u) = \not\square \right\}$$

is a skew two-graph.

Isometries preserve $\Delta(\mathcal{O})$ (or $\nabla(\mathcal{O})$).

Similarities preserve $\Delta(\mathcal{O})$ (or $\nabla(\mathcal{O})$) to within complementation.

Eg. Paley two-graphs Δ_q ($q\equiv 1 \bmod 4$) and skew two-graphs ∇_q ($q\equiv 3 \bmod 4$) from the $Sp(2,q)$ ovoids:

$F = GF(q)$, q odd

$\mathcal{O} = X = PG(1,F)$ projective line

$G = Sp(V,f) = Sp(2,F) \cong SL(2,q)$

acts 2-transitively on X; two orbits on triples $(\langle u\rangle, \langle v\rangle, \langle w\rangle)$, distinguished according to whether or not $f(u,v)f(v,w)f(w,v)$ is a square.

If $q\equiv 1 \bmod 4$ then $\Delta_q := \Delta(X,f)$ is one G-orbit on $\binom{X}{3}$; the other is $\overline{\Delta}_q \cong \Delta_q$.

Regular two-graph of degree $\frac{q-1}{2}$

If $q\equiv 3 \bmod 4$ then $\nabla_q := \nabla(X,f)$ is one G-orbit on $\mathfrak{J}(X)$; the other is $\overline{\nabla}_q \cong \nabla_q$

Taylor (1992) classified the 2-transitive two-graphs.

Theorem Every 2-transitive skew two-graph ∇ is isomorphic to ∇_q (Paley type), some $q \equiv 3 \bmod 4$.

Proof Let $G = \mathrm{Aut}\,\nabla$, and let $g \in G$ involution.
If g fixes $x \in X$ then $(x\,y\,z) \overset{g}{\longleftrightarrow} (x\,z\,y)$ for some $y, z \in X$. But only one of τ, τ^{-1} is in ∇, contradiction. So involutions in G are fixed-point-free.
Bender (1968) $\Rightarrow$ $G \geq PSL(3,q)$, $X = PG(1,q)$, $q \equiv 3 \bmod 4$
$\Rightarrow$ Both orbits of G on X are isomorphic to ∇_q $\square$

Kleidman (1988) classified the 2-transitive ovoids.
If $\mathcal{O}$ is a 2-transitive ovoid then $\Delta(\mathcal{O})$ is 2-transitive or trivial (or $\nabla(\mathcal{O})$ is 2-trans.).

Ovoid	Restrictions	Nontrivial Δ or ∇?	Description
$Sp(2,q)$	$q \equiv 1 \bmod 4$	nontrivial Δ	Paley Δ_q
$Sp(2,q)$	$q \equiv 3 \bmod 4$	nontrivial ∇	Paley ∇_q
$O_3(q)$	q odd	trivial Δ	Theorem 7.2
$U(3,q)$	q odd	nontrivial Δ	unitary two-graph
$O_4^+(q)$	q odd	trivial Δ	$1 + \lfloor \frac{e}{2} \rfloor$ simil. classes
$O_4^-(q)$	q odd	nontrivial Δ	Paley Δ_{q^2}
$O_5(q)$, $O_6^+(q)$	q odd	nontrivial Δ	induced from $O_4^-(q)$
$U(4,q)$	q odd	nontrivial Δ	induced from $U(3,q)$
$O_7(3)$	$q = 3$	nontrivial Δ	$Sp(6,2)$
$O_7(q)$	$q = 3^e$	nontrivial Δ	$PSU(3,q)$
$O_7(q)$	$q = 3^e$, e odd	nontrivial Δ	${}^2G_2(q)$
$O_8^+(q)$	$q = 3^e$	nontrivial Δ	induced from $O_7(q)$

TABLE 1

Our construction of the unitary two-graph from the $U(3,q)$ ovoid follows [Se1]. The nontriviality of $\Delta(\mathcal{O})$ for the last four entries, follows from Theorem 7.4; hence by [Ta2], these are the usual unitary and Ree two-graphs. All remaining cases are covered by remarks above and Theorems 7.2 and 7.3.

Suppose $\mathcal{I}(\mathcal{O})$ is an invariant of caps $\mathcal{O}$ which is computed by testing just k-subsets of $\mathcal{O}$, and that the invariant $\mathcal{I}$ is nontrivial (able to distinguish at least two inequivalent caps of the same size). In the orthogonal case, Theorem 4.3 shows that $k \geq 3$, and that if $k = 3$, then $\mathcal{I}(\mathcal{O})$ is a function of $\Delta(\mathcal{O})$ and the characteristic is odd. [Note: It is usually possible to define a nontrivial invariant graph $\Gamma(\mathcal{O})$; for example, fix $t \geq 0$ and let $\Gamma(\mathcal{O})$ be the set of pairs $\{\langle u \rangle, \langle v \rangle\}$ in $\mathcal{O}$ such that $|\pi \cap \mathcal{O}| = t$ for some plane π of $PG(V)$ containing $\langle u, v \rangle$. However, the latter definition evidently requires testing subsets of $\mathcal{O}$ of size ≥ 4.] For symplectic polar spaces, however, there are nontrivial triple-based invariants in even characteristic, computed by testing for collinear triples (cf. Theorem 4.4).

4.3 Theorem. *Let $\mathcal{P}$ be an orthogonal polar space in $PG(V) = PG(s,F)$, $s \geq 2$, with associated quadratic form Q on V. Then the number of orbits of $P\Omega(V,Q)$ on ordered 3-caps in $\mathcal{P}$ is*

(i) *1, if q is even and s is odd;*
(ii) *2, if q is odd and $\mathcal{P} \neq O^+(4,q)$ (the orbit containing $(\langle u \rangle, \langle v \rangle, \langle w \rangle)$ being determined by whether $f(u,v)f(v,w)f(w,u)$ is a square or a nonsquare in F); or*

$$\underset{\text{cap or ovoid}}{\mathcal{O}} \longrightarrow \underset{\text{or } \nabla(\mathcal{O})}{\Delta(\mathcal{O})} \longrightarrow \underset{\text{fingerprint}}{f(\mathcal{O})}$$

$$\begin{array}{l} \Delta = \Delta(\Gamma) \\ \text{or } \nabla = \nabla(T) \end{array} \longrightarrow \begin{array}{c} A = (0,\pm1)\text{-adjacency} \\ \text{matrix of } \Gamma \text{ or } T \end{array} \longrightarrow \begin{array}{l} f(\Delta) \\ \text{or } f(\nabla) \end{array}$$

$$\left.\begin{array}{l} f(\Delta) \\ f(\nabla) \end{array}\right\} := \text{multiset of entries of } |AA^T|.$$

Theorem If n_λ = no. of $\left\{\begin{array}{c} \{x,y\} \\ (x,y) \end{array}\right\}$ of degree λ in $\left\{\begin{array}{c} \Delta \\ \nabla \end{array}\right\}$

then

$$f(\Delta) = \begin{cases} 0^{2n_{r-1}}\, 2^{2n_{r-2}+2n_r} \cdots (2r-2)^{2n_0+2n_{2r-2}} (2r-1)^{2r}, & v = 2r \\ 1^{2n_{r-1}+2n_r}\, 3^{2n_{r-2}+2n_{r+1}} \cdots (2r-1)^{2n_0+2n_{2r-1}} (2r)^{2r+1}, & v = 2r+1 \end{cases}$$

$$f(\nabla) = \begin{cases} 0^{n_{r-1}}\, 2^{2n_{r-2}} \cdots (2r-2)^{2n_0} (2r-1)^{2r}, & v = 2r \\ 1^{2n_{r-1}}\, 3^{2n_{r-2}} \cdots (2r-1)^{2n_0} (2r)^{2r+1}, & v = 2r+1 \end{cases}$$

Theorem Suppose $\mathcal{P}$ is of orthogonal type, but not $O_4^+(q)$. The number of orbits of $P\Omega(V,Q)$ on caps of size 3 is

1 if q even;

2 if q odd, distinguished by whether or not $f(u,v)f(v,w)f(w,u)$ is a square.

A *partial* m-*system* in $\mathcal{P}$ is a collection $\mathcal{M} = \{\pi_1, \pi_2, \cdots, \pi_k\}$ of m-flats s.t. $\pi_i^\perp \cap \pi_j = \emptyset$ $\forall\, i \neq j$.

Theorem (Shult, Thas) $\exists$ upper bound for $k = |\mathcal{M}|$ depending on $\mathcal{P}$ but not on m.

If $|\mathcal{M}|$ attains this upper bound, $\mathcal{M}$ is an m-*system*.

partial 0-system	$\equiv$	cap
0-system	$\equiv$	ovoid
partial r-system	$\equiv$	partial spread
r-system	$\equiv$	spread

$\{v_{i0}, v_{i1}, \cdots, v_{im}\}$ basis for π_i

$$\operatorname{sgn}(\pi_i, \pi_j) := \operatorname{sgn} \det \left[f(v_{i\alpha}, v_{j\beta}) \right]_{0 \le \alpha, \beta \le m} = \pm 1$$

Theorem If $\mathcal{P}$ is $\left\{\begin{array}{l} \text{orthogonal} \\ \text{unitary} \\ \text{symplectic, } q^{m+1} \not\equiv 3 \bmod 4 \end{array}\right\}$ then

$$\Delta(\mathcal{M}) := \left\{ \{\pi_i, \pi_j, \pi_k\} : \operatorname{sgn}(\pi_i, \pi_j)\operatorname{sgn}(\pi_j, \pi_k)\operatorname{sgn}(\pi_k, \pi_i) = -1 \right\}$$

is a two-graph (trivial if q even)

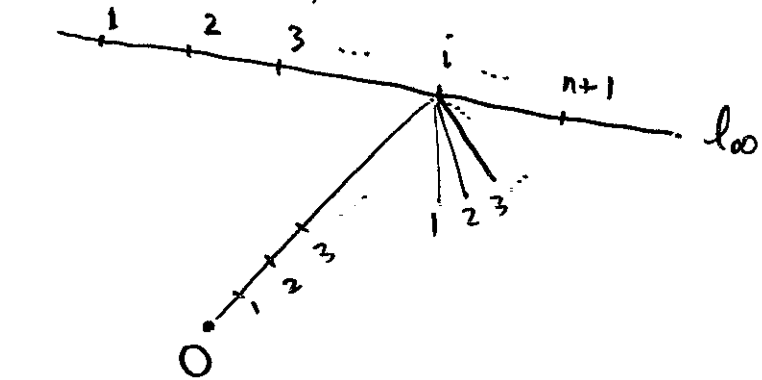
If $\mathcal{P}$ is of symplectic type, $q^{m+1} \equiv 3 \bmod 4$, similarly get a skew two-graph $\nabla(\mathcal{M})$.

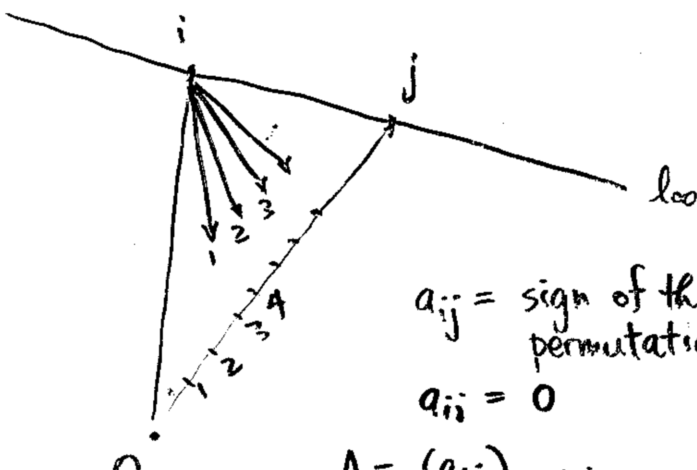
$\Delta(\mathcal{M})$ (or $\nabla(\mathcal{M})$) inveriant under isometries.

$\Delta(\mathcal{M})$ (or $\nabla(\mathcal{M})$) invariant (to within complementation) under similarities.

J.H. Conway's Description

π : translation plane of order n





a_{ij} = sign of this permutation

$a_{ii} = 0$

$A = (a_{ij})_{1 \le i,j \le n+1}$

Fingerprint $f(\pi)$:= multiset of entries of $|AA^T|$.

Theorem Let $\{M_1, M_2, \cdots, M_n\}$ be a spread set for π. Then WLOG

$$a_{ij} = \begin{cases} \operatorname{sgn} \det (M_i - M_j), & 1 \le i,j \le n \\ 1, & i < n+1 = j \\ \pm 1,^* & j < n+1 = j \\ 0, & i = j = n+1 \end{cases}$$

* choose $\begin{cases} +1 & \text{if } n \not\equiv 3 \bmod 4 \\ -1 & \text{if } n \equiv 3 \bmod 4 \end{cases}$

Theorem If $\mathcal{O}$ is an ovoid in $O_6^+(q)$ and π is the corresponding translation plane of order q^2, then $\Delta(\mathcal{O}) = \Delta(\pi)$ to within complementation. So $f(\mathcal{O}) = f(\pi)$.

How large can a cap $\mathcal{O}$ in $\mathcal{P}$ be if $\Delta(\mathcal{O})$ is trivial? What structure of $\mathcal{O}$ attains this maximum?

Theorem $\mathcal{P}$ of type $O_5(q)$, q odd, $\delta = \mathrm{disc}(Q)$. $\mathcal{O}$ cap in $\mathcal{P}$.

(i) $\Delta(\mathcal{O})$ complete and $-2\delta = \square \Rightarrow |\mathcal{O}| \le q+1$

(ii) $\Delta(\mathcal{O})$ empty and $-2\delta = \not\square \Rightarrow |\mathcal{O}| \le q+1$.

Moreover, $\mathcal{O}$ is a BLT-set $\Leftrightarrow$ equality holds in (i) or (ii).

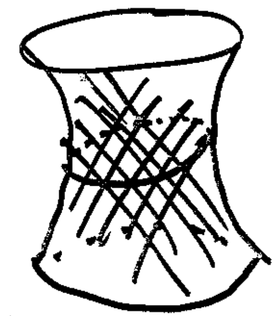
Recall: A BLT-set in $O_5(q)$ is a collection of $q+1$ singular points s.t. $\langle u,v,w\rangle^\perp$ is an elliptic (anisotropic) line $\forall$ distinct $\langle u\rangle, \langle v\rangle, \langle w\rangle$ in $\mathcal{O}$.

BLT-set $\mathcal{O}$ with distinguished point $\longleftrightarrow$ q-clan $\longleftrightarrow$ flock of quadratic cone in $PG(3,q)$

Theorem. Let $\mathcal{O}$ be a $(q+1)$-cap in $O_4^-(q)$, q odd. Then $\Delta(\mathcal{O})$ trivial $\Leftrightarrow$ $\mathcal{O}$ conic.

(BLT $\mathcal{O} \longleftrightarrow$ linear flock)

Theorem $\mathcal{O}$ $(q+1)$-cap in $O_4^+(q)$, $q = p^e$ odd. Then $\Delta(\mathcal{O})$ trivial $\Leftrightarrow$ $\mathcal{O}$ 2-transitive



(e iso. classes, $1 + \lfloor \frac{e}{2} \rfloor$ simil. classes)

(BLT $\mathcal{O} \longleftrightarrow$ linear or Kantor flock)

Proofs use Blokhuis (1984) and Carlitz (1960).

These results are equivalent to Thas (1987) in flock language.

Theorem $\mathcal{P}$ polar space nat. embedded in $PG(n,q)$, $q=p^e$ odd.
$\mathcal{O}$ cap in $\mathcal{P}$.

$$\Delta(\mathcal{O}) \text{ trivial} \Rightarrow |\mathcal{O}| \leq \binom{n+\frac{p-1}{2}}{n}^e + 1.$$

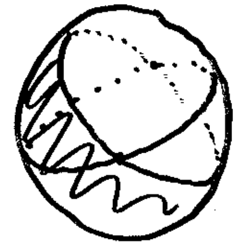
$$\begin{array}{c|c|}
 & \varphi \in V^* \\
\hline
v \in V & \cdots (\varphi(v))^{q-1} \\
\hline
\end{array}$$

has p-rank $\binom{n+p-1}{n}^e$

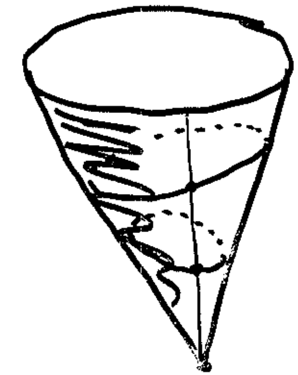
$$\begin{array}{c|c|}
 & \varphi \in V^* \\
\hline
v \in V & \cdots \varphi(v)^{\frac{q-1}{2}} \\
\hline
\end{array}$$

has p-rank $\binom{n+\frac{p-1}{2}}{n}^e$.

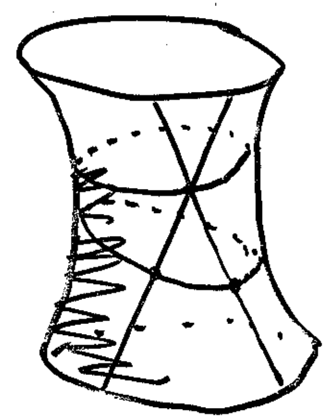
Classical Circle Geometries



Möbius (inversive) plane



Laguerre plane



Minkowski plane

$\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ doubly intersecting circles in classical Laguerre plane, i.e. $|C_i \cap C_j| = 2 \quad \forall i \neq j$

Theorem $\exists$ family of $(3q-1)/2$ doubly intersecting circles.

Theorem $k = |\mathcal{C}| \leq \frac{q^2+1}{2}$

(cf. Blokhuis and Bruen (1989) for Miquelian inversive plane)

Theorem $|\mathcal{C}| \leq \binom{(p+7)/2}{4}^e$ where $q = p^e$ odd.

$$\Rightarrow |\mathcal{C}| \leq \begin{cases} q^{1.47}, & p = 3 \\ q^{1.68}, & p = 5 \\ q^{1.83}, & p = 7 \end{cases}$$