Transfer Matrix Method

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Reference: Transfer Matrix Method

I.M. Gessel and R.P. Stanley, 'Algebraic Enumeration', in *Handbook of Combinatorics Vol. 2*, ed. R.L. Graham et al., Elsevier, 1995, pp.1021–1061.

References: Dimensions of Codes

N. Hamada, 'The rank of the incidence matrix of points and d-flats in finite geometries', J. Sci. Hiroshima Univ. Ser. A-I **32** (1968), 381–396.

M. Bardoe and P. Sin, 'The permutation modules for $GL(n+1,q)$ acting on $P^n(q)$ and F_q^{n+1}', to appear in JLMS.

`http://www.math.ufl.edu/~sin/preprints/hamada.dvi`

G.E. Moorhouse, 'Dimensions of Codes from Finite Projective Spaces' (as html and as Maple worksheet)

`http://math.uwyo.edu/~moorhous/src/hamada.html`

`http://math.uwyo.edu/~moorhous/src/hamada.mws`

Problem 1

Let S_k be the set of 'words' of length k consisting of 'a's and 'b's, with no two consecutive 'b's. Determine $F_k = |S_k|$.

$F_0 = 1$	$F_1 = 2$	$F_2 = 3$	$F_3 = 5$	$F_4 = 8$
''	'a'	'aa'	'aaa'	'aaaa'
	'b'	'ab'	'aab'	'aaab'
		'ba'	'aba'	'aaba'
			'baa'	'abaa'
			'bab'	'abab'
				'baaa'
				'baab'
				'baba'

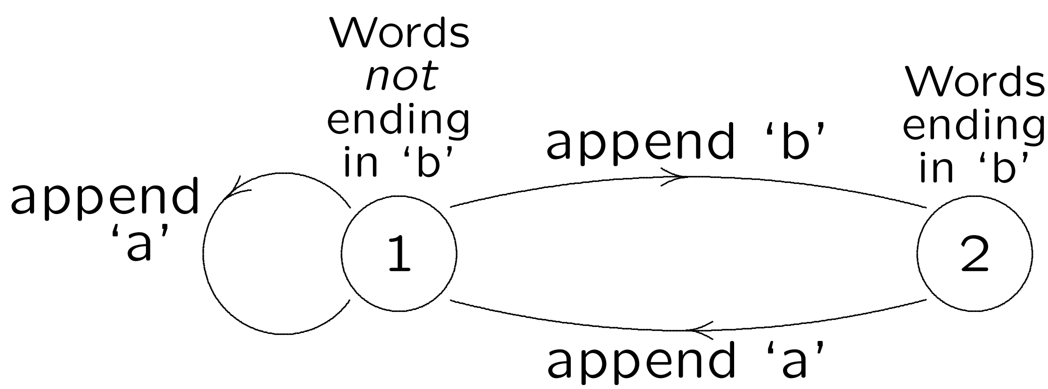
etc. This gives all but the first term of the *Fibonacci sequence*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

To find a formula for F_k, we work instead with the *generating function*

$$\sum_{k=0}^{\infty} F_k t^k = 1 + 2t + 3t^2 + 5t^3 + 8t^4 + 13t^5 + \cdots$$

Observe that words $w \in S_k$ correspond to paths of length k, starting at vertex 1 in the digraph



Agenda

1. Motivating Problem 1 (above)
2. Counting Walks by the Transfer Matrix Method
3. Application to Problem 1
4. Counting Closed Walks
5. Counting Weighted Walks in Digraphs with Weighted Edges
6. MAPLE Worksheet for Problem 1
7. Application to Coding Theory

The Transfer Matrix Method

Let D be a digraph (directed graph), possibly with loops, having vertices 1,2,3,. . . ,n. Let $A = [a_{ij} \ : \ 1 \leq i,j \leq n]$ be the adjacency matrix of D; in other words,

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge of } D; \\ 0, & \text{otherwise.} \end{cases}$$

A **walk of length** k in D is a sequence

$$i_0 \longrightarrow i_1 \longrightarrow i_2 \longrightarrow \quad \cdots \quad \longrightarrow i_k$$

of (not necessarily distinct) vertices such that each $i_{r-1} \longrightarrow i_r$ is an edge of D.

Counting Walks from i to j

Let $w_{ij}(k)$ be the number of walks of length k from vertex i to vertex j in D. Then $w_{ij}(k)$ is the (i,j)-entry of A^k. This is readily computed by reading off the coefficient of t^k in the generating function $\sum_{k\geq 0} w_{ij}(k)t^k$ which in turn is the (i,j)-entry of

$$(I-tA)^{-1} = I + tA + t^2A^2 + t^3A^3 + \cdots.$$

Since the (i,j)-entry of $(I-tA)^{-1}$ is of the form

$$\frac{\text{poly. in } t \text{ of degree} \leq n-1}{\det(I-tA)},$$

$w_{ij}(k)$ satisfies a linear recurrence

$$w_{ij}(k+n) = \sum_{r=0}^{n-1} c_r w_{ij}(k+r) \quad \text{for all } k \geq 0$$

where $\det(I-tA) = 1 - c_{n-1}t - c_{n-2}t^2 - \cdots - c_0t^n$. The initial conditions $w_{ij}(0)$, $w_{ij}(1)$, ..., $w_{ij}(n-1)$ depend on i and j but the recurrence does not.

Counting All Walks

Let $w(k) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}(k)$, the total number of walks of length k. This is the coefficient of t^k in the *sum* of the entries of $(I - tA)^{-1}$.

In particular $w(k)$ satisfies the same recurrence as the $w_{ij}(k)$'s:

$$w(k+n) = \sum_{r=0}^{n-1} c_r w(k+r) \quad \text{for all } k \geq 0$$

but with different initial conditions.

Counting Closed Walks

Let $w_{closed}(k) = \sum_{i=1}^{n} w_{ii}(k)$, the total number of *closed* walks of length k (i.e. starting and ending at the same vertex). This is the coefficient of t^k in $trace((I - tA)^{-1})$.

In particular $w_{closed}(k)$ satisfies the same linear recurrence as the $w_{ij}(k)$'s and $w(k)$, but again with different initial conditions.

Here we assumed the initial/final vertex to be distinguished, i.e. the walks $(i_0, i_1, i_2, \ldots, i_k)$ and $(i_1, i_2, \ldots, i_k, i_0)$ are counted as distinct unless all $i_0 = i_1 = \cdots = i_k$.

Example

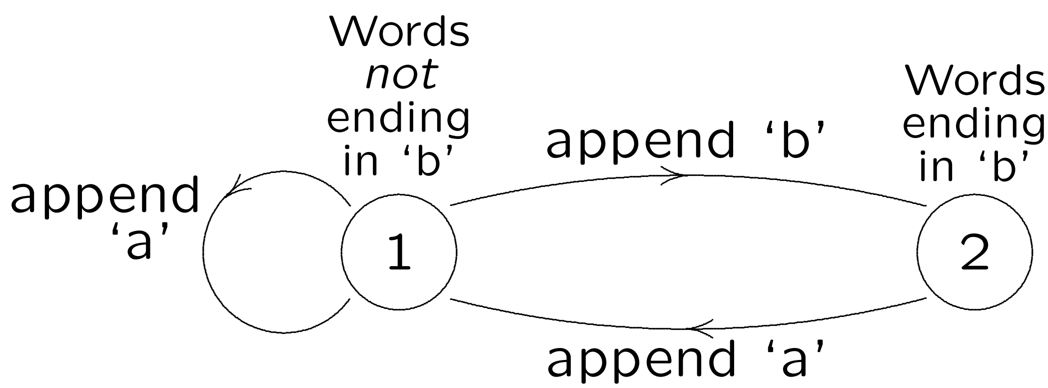
Let F_k be the number of 'words' of length k consisting of 'a's and 'b's, with no two consecutive 'b's.

$F_0 = 1$	$F_1 = 2$	$F_2 = 3$	$F_3 = 5$	$F_4 = 8$
''	'a'	'aa'	'aaa'	'aaaa'
	'b'	'ab'	'aab'	'aaab'
		'ba'	'aba'	'aaba'
			'baa'	'abaa'
			'bab'	'abab'
				'baaa'
				'baab'
				'baba'

etc. This gives all but the first term of the *Fibonacci sequence*

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Observe that F_k is the number of paths of length k, starting at vertex 1 in the digraph



$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(I - tA)^{-1} = \frac{1}{1-t-t^2} \begin{bmatrix} 1 & t \\ t & 1-t \end{bmatrix}$$

$$
\begin{aligned}
\sum_{k\geq 0} F_k t^k &= \text{sum of } (1,1)\text{- and } (1,2)\text{-} \\
&\quad \text{entries of } (I-tA)^{-1} \\
&= \frac{1+t}{1-t-t^2} \\
&= \frac{1}{\sqrt{5}}\Big(\frac{\alpha^2}{1-\alpha t} - \frac{\beta^2}{1-\beta t}\Big) \\
&= \frac{1}{\sqrt{5}} \sum_{k\geq 0} (\alpha^{k+2} - \beta^{k+2}) t^k
\end{aligned}
$$

where $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$. From $(1-t-t^2)\sum_{k\geq 0} F_k t^k = 1+t$ we obtain

$$
F_k = \begin{cases} 1, & \text{if } k = 0; \\ 2, & \text{if } k = 1; \\ F_{k-1} + F_{k-2}, & \text{if } k \geq 2 \end{cases}
$$

so by induction, F_k is the $(k+1)$st Fibonacci number. From the series expansion we obtain the explicit formula

$$
F_k = \frac{\alpha^{k+2} - \beta^{k+2}}{\sqrt{5}} \qquad \text{for } k \geq 0.
$$

Wraparound Version

Let L_k (for $k \geq 0$) be the number of 'words' of length k consisting of 'a's and 'b's with no consecutive 'b's, and which do not both start and end with 'b'. For technical reasons we will take $L_0 = 2$.

For $k \geq 2$, we are simply counting necklaces with **a**mber and **b**lack beads having no two consecutive black beads; however, each necklace has a distinguished starting point (a knot in its cord) and a distinguished direction (clockwise or counter-clockwise).

$L_1 = 1$	$L_2 = 3$	$L_3 = 4$	$L_4 = 7$
'a'	'aa'	'aaa'	'aaaa'
	'ab'	'aab'	'aaab'
	'ba'	'aba'	'aaba'
		'baa'	'abaa'
			'abab'
			'baaa'
			'baba'

These are the familiar *Lucas numbers* which satisfy the same recurrence relation as the Fibonacci numbers, but a different initial condition.

Note that L_k is the number of closed walks of length k in our digraph.

$$\begin{aligned}\sum_{k\geq 0} L_k t^k &= trace((I-tA)^{-1}) \\ &= \frac{2-t}{1-t-t^2} \\ &= \frac{1}{1-\alpha t}+\frac{1}{1-\beta t} \\ &= \sum_{k\geq 0}(\alpha^k+\beta^k)t^k\end{aligned}$$

From $(1-t-t^2)\sum_{k\geq 0} L_k t^k = 2-t$ we obtain

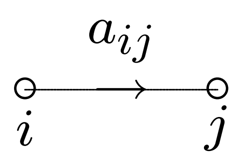
$$L_k = \begin{cases} 2, & \text{if } k=0; \\ 1, & \text{if } k=1; \\ L_{k-1}+L_{k-2}, & \text{if } k\geq 2 \end{cases}$$

From the series expansion we obtain the explicit formula

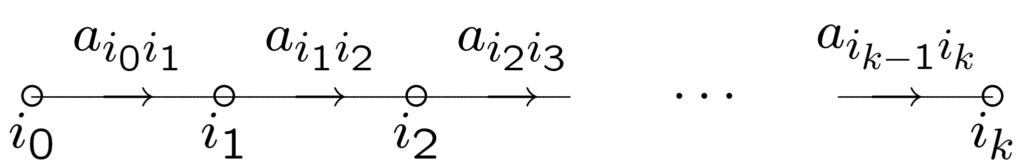
$$L_k = \alpha^k + \beta^k \qquad \text{for } k \geq 0.$$

Counting Walks with Weighted Edges

As before, D is a digraph (directed graph), possibly with loops, having vertices 1, 2, 3, . . . , n. Assign a weight to each edge:



(Non-edges have weight zero.) Define the **weight** of a walk



of length k to be the product

$$a_{i_0 i_1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}.$$

Let $A = [a_{ij} \ : \ 1 \le i, j \le n]$.

Then

$$w_{ij}(k) \quad := \begin{pmatrix} \text{The sum of all weights} \\ \text{of walks in } D \text{ of length } k \\ \text{from vertex } i \text{ to vertex } j \end{pmatrix}$$

$$= (i,j)\text{-entry of } A^k$$

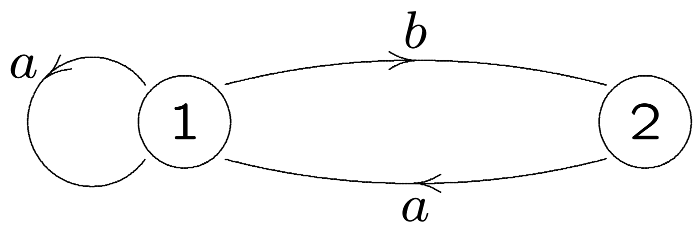
has generating function $\sum_{k\geq 0} w_{ij}(k)t^k$ equal to the (i,j)-entry of

$$(I - tA)^{-1} = I + tA + t^2A^2 + t^3A^3 + \cdots$$

as before.

Example

We have determined the number F_k of words of length k consisting of 'a's and 'b's, with no two consecutive 'b's. How many such words contain r 'a's and (therefore) $k-r$ 'b's?



$$A = \begin{bmatrix} a & b \\ a & 0 \end{bmatrix}$$

$$(I - tA)^{-1} = \frac{1}{1{-}at{-}abt^2} \begin{bmatrix} 1 & bt \\ at & 1{-}at \end{bmatrix}$$

The sum of the $(1,1)$- and $(1,2)$-entries is

$$\frac{1+bt}{1{-}at{-}abt^2} = 1 + (a{+}b)t + (a^2{+}2ab)t^2 + (a^3{+}3a^2b{+}ab^2)t^3 + (a^4{+}4a^3b{+}3a^2b^2)t^4 + \cdots$$

Thus, for example, among the $F_4{=}8$ words of length 4,

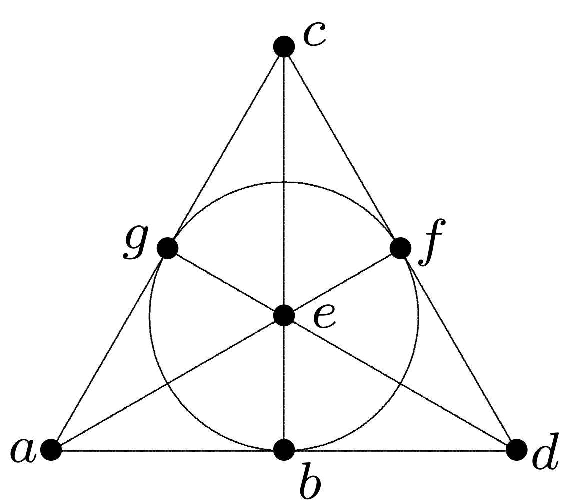
1 has 4 ‘a’s and 0 ‘b’s;

4 have 3 ‘a’s and 1 ‘b’;

3 have 2 ‘a’s and 2 ‘b’s.

Codes from Finite Geometry

Consider the *projective plane of order 2*:



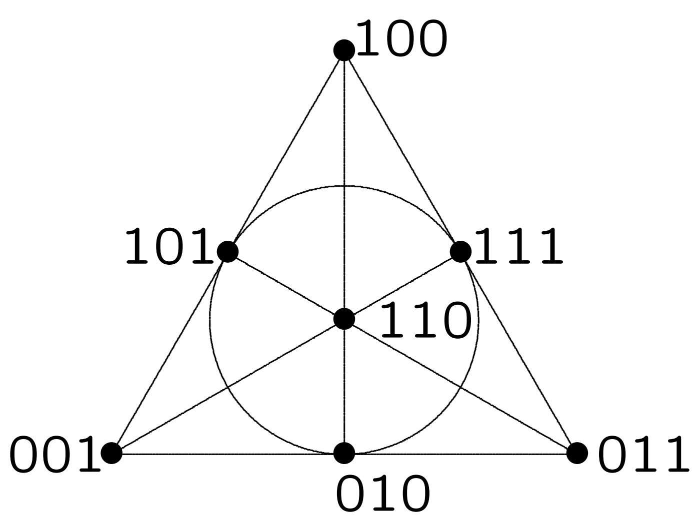
The *binary code* of this geometry is the subspace $\mathcal{C} \le F^7$ (where $F = \{0,1\}$ mod 2) spanned by the lines:

$$\begin{array}{rll}
 & \scriptstyle a\,b\,c\,d\,e\,f\,g & \scriptstyle a\,b\,c\,d\,e\,f\,g \\
\mathcal{C} = \{ & 0000000, & 1111111, \\
 & 1101000, & 0010111, \\
 & 0110100, & 1001011, \\
 & 0011010, & 1100101, \\
 & 0001101, & 1110010, \\
 & 1000110, & 0111001, \\
 & 0100011, & 1011100, \\
 & 1010001, & 0101110\}
\end{array}$$

$$|\mathcal{C}| = 2^4; \qquad \dim \mathcal{C} = 4$$

The code above is the *1-error correcting binary Hamming code of length 7.*

The projective plane is constructed from F^3 by taking as points and lines the 1- and 2-dimensional subspaces of F^3.



Codes of Finite Projective Spaces

Let F be the field of order p^e, p prime. *Projective n-space over F* has as its points, lines, etc. the subspaces of F^{n+1} of dimension 1, 2, etc.

Problem: Compute the dimension of the code $\mathcal{C} = \mathcal{C}_{n,p,e,k}$ spanned by the subspaces of codimension k.

Solution by Hamada's Formula (the following theorem) is usually computationally infeasible.

Solution by the Transfer Matrix Method

Theorem (Bardoe and Sin, 1999)
Define $M(t) = (1 + t + t^2 + \cdots + t^{p-1})^{n+1}$.

Let $D = D_{n,p,e,k}$ be the digraph with vertices $1, 2, \ldots, k$, and the edge from vertex i to vertex j has weight equal to the coefficient of t^{pj-i} in $M(t)$. Then

$$\dim \mathcal{C}_{n,p,e,k} = 1 + \begin{pmatrix} \text{sum of weights} \\ \text{of closed walks} \\ \text{of length } e \\ \text{in } D \end{pmatrix}$$

$$= 1 + \begin{pmatrix} \text{coeff. of } t^e \\ \text{in } tr[(I - tA)^{-1}] \end{pmatrix}$$

where A is the $k \times k$ matrix whose (i, j)-entry is the weight of edge (i, j) (defined above).

Example: Projective Plane of Order 2

$\mathcal{C}$ = binary code spanned by the seven lines (subspaces of codimension $k = 1$)

$M(t) = (1 + t)^3 = 1 + 3t + 3t^2 + t^3$

$A = [3]$ (coefficient of t^1 in $M(t)$)

$(I - tA)^{-1} = \left[\frac{1}{1-3t}\right]$

$tr[(I - tA)^{-1}] = \frac{1}{1-3t} = 1 + 3t + 9t^2 + 27t^3 + \cdots$

$\dim \mathcal{C} = 1 + 3 = 4$