Finite Projective Planes and their Substructures

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Planes

A *projective plane of order* $n \geqslant 2$ is an incidence structure consisting of $n^2 + n + 1$ points and the same number of lines, such that

- every line contains exactly $n + 1$ points;
- every point lies on exactly $n + 1$ lines; and
- every pair of distinct points is joined by a unique line.

Known planes of small order

Number of planes up to isomorphism (i.e. collineations):

n	number of planes of order n
2	1
3	1
4	1
5	1
7	1
8	1
9	4
11	$\geqslant 1$
13	$\geqslant 1$

n	number of planes of order n
16	$\geqslant 22$
17	$\geqslant 1$
19	$\geqslant 1$
23	$\geqslant 1$
25	$\geqslant 193$
27	$\geqslant 13$
29	$\geqslant 1$
$\dots$	$\dots$
49	$> 10^5$

`pzip`: A compression utility for finite planes

Storage requirements for a projective plane of order *n*:

n	size of line sets	size of MOLS	`gzipped` MOLS	`pzip`
11	5 KB	1.3 KB	0.2 KB	0.06 KB
25	63 KB	15 KB	9 KB	0.9 KB
49	550 KB	110 KB	81 KB	6 KB

See http://www.uwyo.edu/moorhouse/pzip.html

Open Questions

Does every finite projective plane have prime power order?

Is every projective plane of prime order classical?

Find a rigid finite projective plane (i.e. one with no nontrivial collineations).

Does every projective plane of order n^2 contain a subplane of order n? and a unital of order n?

Does every nonclassical finite projective plane have a subplane of order 2? ('Neumann's Conjecture')

Does every finite partial linear space embed in a finite projective plane?

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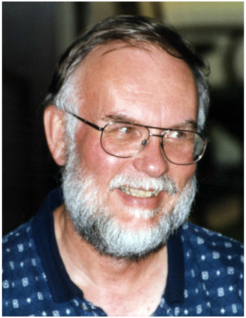

"The survival of finite geometry as an active field of study probably depends on someone finding a finite projective plane of non-prime-power order."

—Gary Ebert

What approach to searching for a non-prime-power order plane offers the greatest hope for success?

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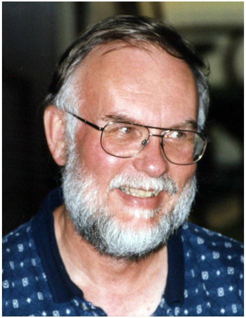

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Subplanes of known planes

Tim's suggestion: Consider known planes (there is an ample supply). Generate subplanes (this is not hard). Check to see if any nonclassical subplanes arise (this is also easy).

Subplanes of known planes of small order

Among the 193 known planes of order 25,

- all have subplanes of order 5;
- all except the classical plane have subplanes of order 2;
- only a very few have subplanes of order 3 (the ordinary Hughes plane and six closely related planes);
- no other orders of subplanes arise.

The number of subplanes of each order is listed at http://www.uwyo.edu/moorhouse/pub/planes25/

Subplanes of known planes of small order

Among the hundreds of thousands of known planes of order 49,

- all have subplanes of order 7;
- all except the classical plane have subplanes of order 2;
- *very few* have subplanes of order 3 (about 1 in every 20,000 planes);
- no other orders of subplanes arise.

Heuristic number of subplanes of order 2

Let Π be a 'randomly chosen' plane of order n.

Let $N_k(\Pi)$ the number of subplanes of order k.

Heuristically,

$$N_2(\Pi) \approx \frac{1}{168} n^3(n^3-1)(n+1) \sim \frac{n^7}{168}$$

Why? (A *back-of-the envelope* estimate only):

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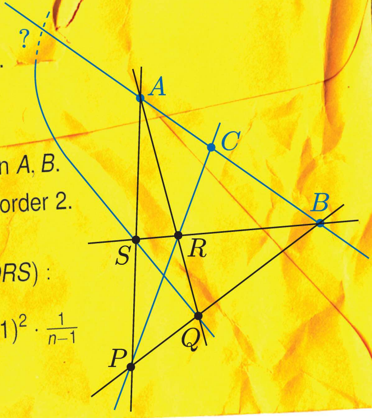
Let $PQRS$ be a randomly chosen quadrangle in Π.



Construct A, B, C as shown.

QS meets AB at any of the $n-1$ points other than A, B.

Only one choice (namely, C) gives a subplane of order 2.

Number of 5-tuples
(P, Q, R, S, subplane of order 2 generated by $PQRS$) :

$$168 N_2(\Pi) \approx (n^2+n+1)(n^2+n)(n^2)(n-1)^2 \cdot \frac{1}{n-1}$$

$$N_2(\Pi) \approx \frac{1}{168} n^3(n^3-1)(n+1)$$

Heuristic number of subplanes of order 2

How good is this estimate?

For $n = 25$ it predicts 37,781,250 subplanes of order 2.

Ignoring the translation planes and two Hughes planes, the actual number of subplanes of order 2 varies from

35,110,000 to 43,569,000.

For $n = 49$ the estimate is much better.

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Let Π be a ‘randomly chosen’ plane of order n.

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One might guess (naïvely) that in general, larger planes might have *more* subplanes of small order k.

This is only true for $k = 2$! Heuristically,

$$N_k(\Pi) \sim c_k n^{(3-k)(k^2+k+1)}$$

as $n \to \infty$, where c_k is a constant depending only on k. For example,

$$N_3(\Pi) = O(1)$$

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The heuristic

$$N_k(\Pi) \sim c_k n^{(3-k)(k^2+k+1)}$$

only applies for fixed k as $n \to \infty$. It fails for counting Baer subplanes $k = \sqrt{n}$.

Question: What is a reasonable heuristic for estimating the number of Baer subplanes?

Neumann's Conjecture and Goldbach's Conjecture

Goldbach's conjecture:

"Every even number $\geqslant 4$ is a sum of two primes"

is 'almost certainly true' based on some reasonable heuristics regarding the distribution of primes.

Neumann's Conjecture and Goldbach's Conjecture

Neumann's conjecture:

"Every nondesarguesian plane of order n contains subplanes of order 2"

seems likely to be true for similar reasons.

Denote by $P(n)$ the number of isomorphism classes of projective planes of order n.

Heuristically the number of planes of order n without subplanes of order 2 is about

$$\frac{P(n)}{e^{N_2}} \approx \frac{P(n)}{e^{n^7/168}}.$$

This tends to zero unless $P(n)$ grows *very* fast.

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Subplanes of order 3 in ordinary Hughes planes

Theorem (Caliskan and M., 2010)

Let $q \equiv 5 \bmod 6$. The ordinary Hughes plane of order q^2 has subplanes of order 3.

Computational results suggest that for $q \equiv 1$ mod 6, Hughes planes of order q^2 have no subplanes of order 3.

(For $q = 3^e$, subplanes of order 3 exist trivially.)

More General Substructures

A *partial linear space* is a point-line incidence system in which any two lines meet in at most one point. (Further technical requirement: each line has at least 2 points.)

Does every finite partial linear space embed in a *finite* projective plane?

Theorem (Caliskan and M., 2010)

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Nets

Let $k \in \{1, 2, 3, \ldots, n+1\}$.

A *k-net of order n* is an incidence structure consisting of n^2 points and nk lines with

- n points on each line;
- lines partitioned into k parallel classes of n lines each;
- for two lines $\ell \neq m$,

$$|\ell \cap m| = \begin{cases} 0, & \text{if } \ell || m; \\ 1, & \text{otherwise.} \end{cases}$$

For $k = n + 1$ this gives an affine plane.

For $k \geqslant 3$, a k-net of order n may be specified using $k - 2$ MOLS(n) (that's $k - 2$ mutually orthogonal Latin squares of order n).

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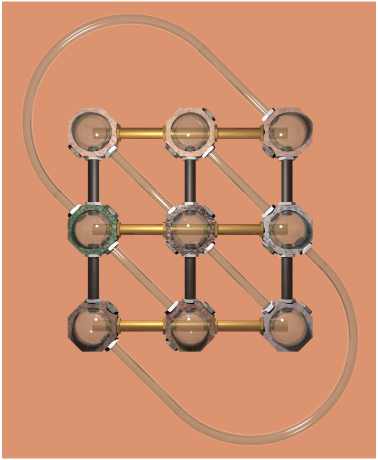
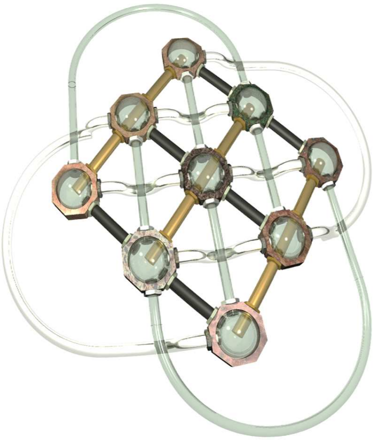


3-net of order 3



4-net of order 3

Ranks of nets

The *p-rank* of an incidence structure is the rank of its $(0,1)$-incidence matrix over a field of characteristic p.

In 1991, I conjectured:

Consider a k-net $\mathcal{N}_k$ of order n. Let p be a prime sharply dividing n, i.e. $p \mid n$ but $p^2 \nmid n$. Let $\mathcal{N}_{k-1}$ be a $(k-1)$-subnet formed by deleting the lines of one parallel class (chosen arbitrarily). Then

$$\mathrm{rank}_p(\mathcal{N}_k) - \mathrm{rank}_p(\mathcal{N}_{k-1}) \geqslant n - k + 1.$$

This would imply that any projective plane of squarefree order, or of order $n \equiv 2 \bmod 4$, is in fact classical of prime order.

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Howard-Myrvold 4-net of order 10

Leah Howard and Wendy Myrvold (U. Victoria, 2009) found the following 4-net of order 10:

00	67	58	49	83	72	91	16	25	34
76	11	84	57	29	93	08	35	60	42
85	48	22	96	71	09	37	63	14	50
94	75	69	33	07	18	82	40	51	26
38	92	17	70	44	86	59	21	03	65
27	39	90	81	68	55	74	02	46	13
19	80	73	28	95	47	66	54	32	01
61	53	36	04	12	20	45	77	88	99
43	24	05	62	56	31	10	89	97	78
52	06	41	15	30	64	23	98	79	87

Its 2-rank is 34, but every 3-subnet has 2-rank equal to 28. It does not extend to a 5-net.

This is the first known counterexample to my rank conjecture for nets.

Reconsidering the rank conjecture

The rank conjecture is still open for $n = p$ prime.

The rank conjecture is also open under the additional hypothesis that the nets extend to affine planes.

This is the most important case (for the application to projective planes, this is the only case we care about).

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The Plight of Theoretical Physics

The biggest questions in theoretical physics may *never* be answered due to our limited resources (in particular the massive amounts of energy required to test current theories).

How do the open questions in finite geometry compare?

We cannot rule out the possibility of a new idea.

Failing that, it would be very useful to have better heuristics for gauging the computational difficulty of the biggest open problems.

Where heuristic counts would be useful

About how large should n be in order to have a rigid plane of order n?

Roughly how large should n be to have a projective plane of non-prime-power order?

Roughly how many Hadamard matrices of order $4n$ should there be? Do we expect Hadamard matrices of order $4n$ to exist for all n?

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