Construction of Some New Projective Planes

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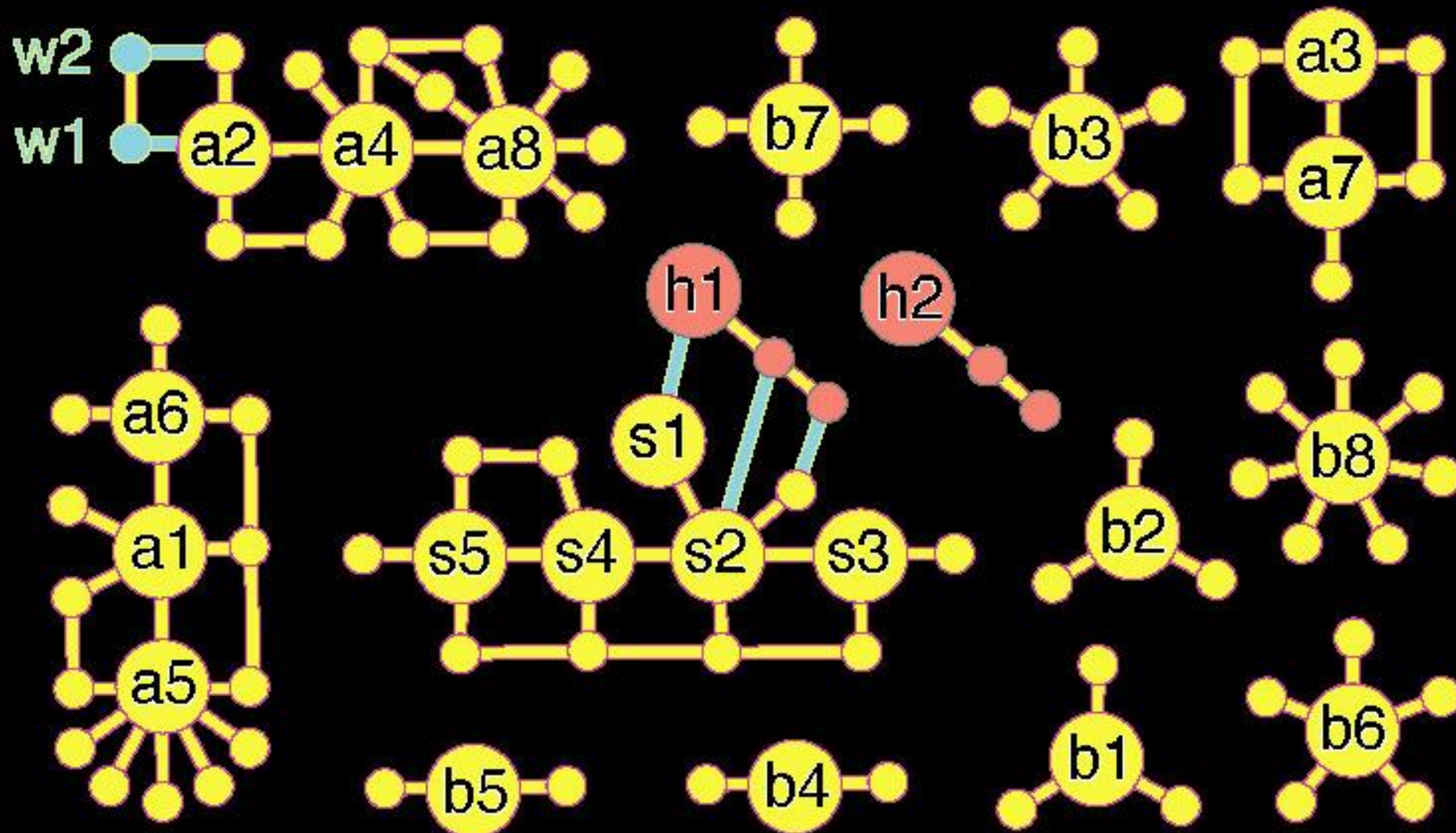
http://math.uwyo.edu/~moorhous/pub/planes/

n	2	3	4	5	7	8	9	11
number of planes of order n known	1	1	1	1	1	1	4	1

n	13	16	17	19	23	25	27	29
number of planes of order n known	1	22	1	1	1	193	13	1

Known Planes of Order 25



Translation planes a1,…,a8; b1,…,b8; s1,…,s5 classified by Czerwinski & Oakden (1992)


The Wyoming Plains

$|Aut(w1)| = 19200$

$|Aut(w2)| = 3200$

The Wyoming Planes

Thanks to my coauthor…

Thanks to `nauty` and `GAP` !

I make extensive use of Brendan McKay's software package `nauty`; also `GAP` for group computations.

Given a graph Γ, `nauty` will determine

- the automorphism group of Γ, and
- a 'canonical' representative of the isomorphism class of Γ. (Thus $\Gamma \cong \Gamma'$ iff the graphs Γ and Γ' have the same canonical representative.)

This can be applied to the *incidence graph* Γ_Π of a projective plane Π of order n. This is a bipartite graph with $2(n^2+n+1)$ vertices (one for each point/line of Π) and edges corresponding to incident point-line pairs. Note that $\mathrm{Aut}(\Gamma_\Pi)$ is the group of all *collineations* and *correlations* of Π. If desired, `nauty` can preserve the two parts of the partition, thereby obtaining just the collineations of Π.

Limitations of `nauty`

Projective planes are time-consuming cases for **`nauty`**.

Planes of order 16 require minutes with Gordon Royle's invariant `cellfano2`). Planes of order 25 or 27 require hours or days. Planes of order 32 are infeasible.

A new idea is needed!

Better Approach: ‘Conway Doubling’

Let Π be a projective plane of order n. We define a graph Δ_Π with $4(n^2+n+1)$ vertices (*roughly* a double cover of the *non*-incidence graph Γ_Π) such that

- $\Delta_\Pi \cong \Delta_{\Pi'}$ iff $\Gamma_\Pi \cong \Gamma_{\Pi'}$ iff $\Pi \cong \Pi'$;
- Aut(Δ_Π) is usually much faster to compute than Aut(Π), and
- $\mathrm{Aut}(\Pi) = \mathrm{Aut}(\Delta_\Pi)/Z$ (where $Z \subseteq Z(\mathrm{Aut}(\Delta_\Pi))$) is easily obtained.

Definition of Δ_Π

Let $F = \{0, 1\}$ be the field of order two. Vertices of Δ_Π are of the form $(P, i), (L, j)$ where P and L are a point and line of Π, and $i, j \in F$. Adjacency ...

Index the points on each line using $0, 1, 2, \ldots, n$ using a fixed (but arbitrary) ordering. Index the lines through each point using $0, 1, 2, \ldots, n$. For each non-incident P, L in Π, we obtain a permutation $\sigma_{P,L}$ of the symbols $0, 1, 2, \ldots, n$.

There are two types of edges in Δ_Π:

(I) $(P, i) \sim (L, j)$ iff

$P \notin L$ and $\mathrm{sgn}(\sigma_{P,L}) = (-1)^{i+j}$

(II) $(P, 0) \sim (P, 1), \quad (L, 0) \sim (L, 1)$

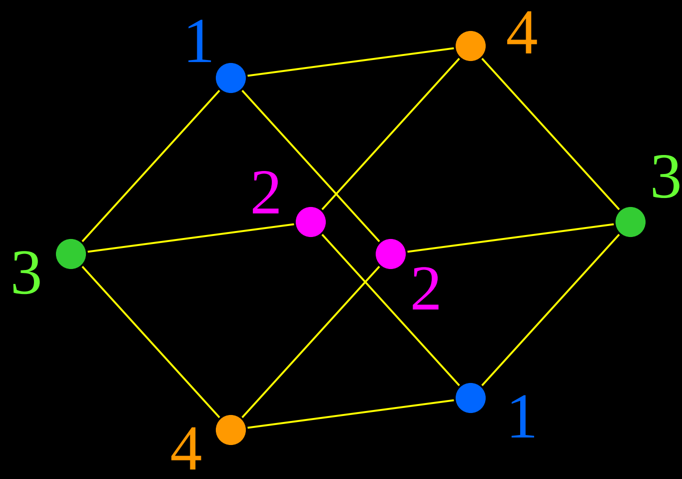
Type (I) edges form a double cover of Γ_Π. Type (II) edges ensure that Δ_Π is connected; without them, Δ_Π could be two disjoint copies of Γ_Π, with automorphism group $\mathrm{Aut}(\Pi) \wr 2$. (This happens if Π is a classical plane of even order.)

$Z \subseteq Z(\mathsf{Aut}(\Delta_\Pi))$ is the subgroup of order two generated by

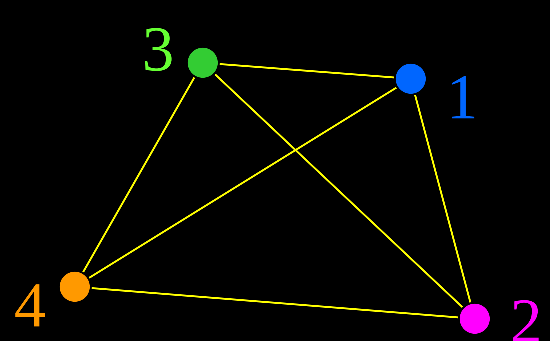
$$(P, 0) \leftrightarrow (P, 1), \quad (L, 0) \leftrightarrow (L, 1).$$

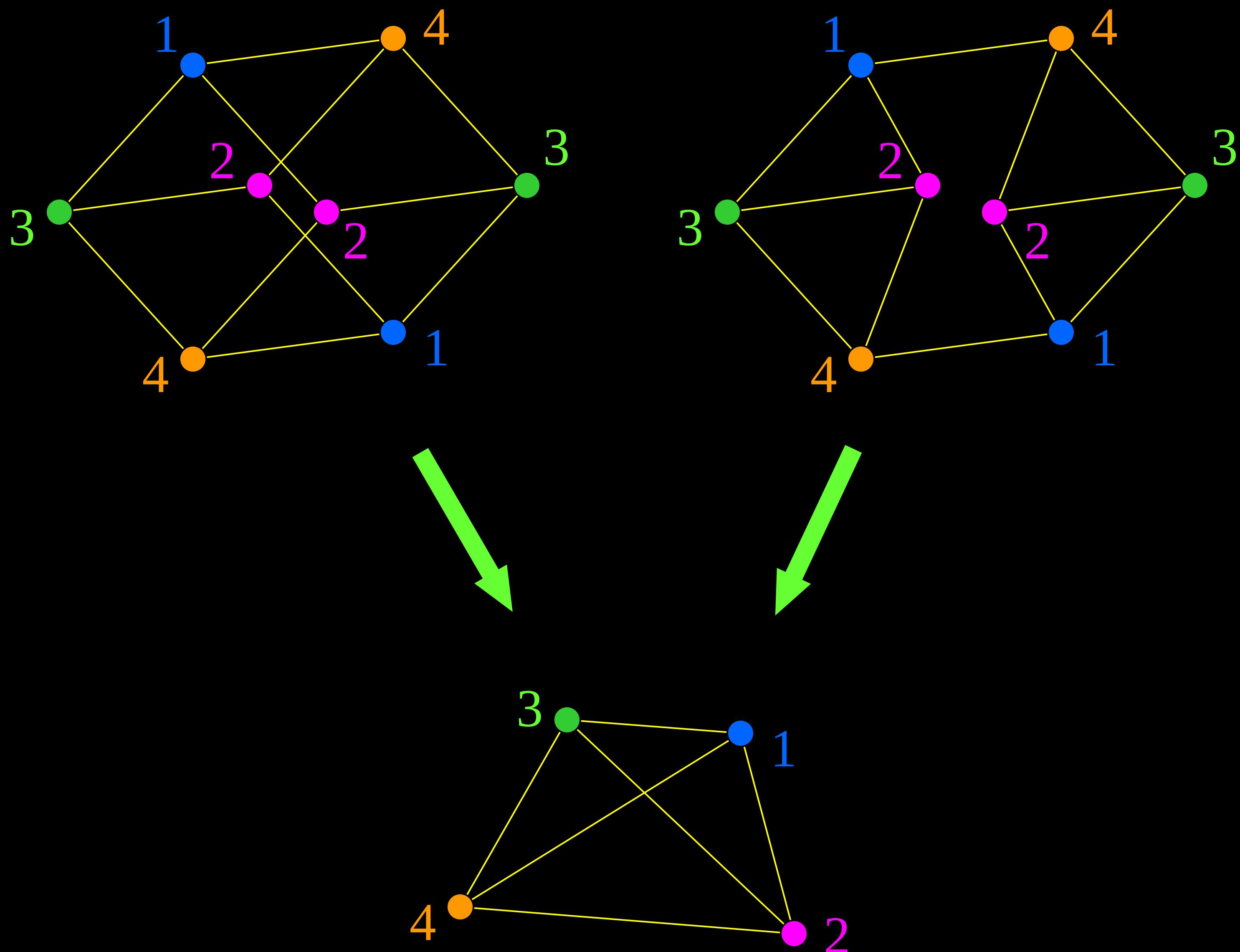
Theorem. $Aut(\Delta_\Pi)/Z = \mathsf{Aut}(\Gamma_\Pi) = \mathsf{Aut}(\Pi)$.

Where do the new planes come from?



quotient by τ, an automorphism of order 2




1
4
3
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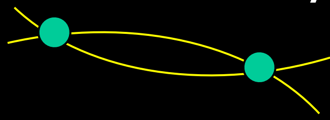
2

Given a projective plane P with involution $\tau \in \mathrm{Aut}(\Pi)$,

let Π/τ be the incidence structure induced on point and line orbits of size 2.

Π/τ yields a cell complex Δ having

- vertices (0-*cochains*): points, blocks of Π/τ
- edges (1-*cochains*): flags of Π/τ
- square faces (2-*cochains):* “digons” of Π/τ



$C^i = C^i(\Delta, \mathbb{F}_2) = \mathbb{F}_2$-space of i-cochains

coboundary map $C^0 \overset{\delta^0}{\rightarrow} C^1 \overset{\delta^1}{\rightarrow} C^2$

$H^1 = H^1(\Delta, \mathbb{F}_2) = \ker \delta^1 / \mathrm{im}\ \delta^0$

If $H^1 = 0$ then Π/τ lifts uniquely back to Π.

Theorem.

Equivalence classes of pairs (Π,τ) covering the same Π/τ $\Longleftrightarrow$ orbits of $\mathrm{Aut}(\Pi/\tau)$ on $H^1(\Delta,\mathbb{F}_2)$

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actually, a particular coset of $H^1=Z^1/B^1$ in C^1/B^1…

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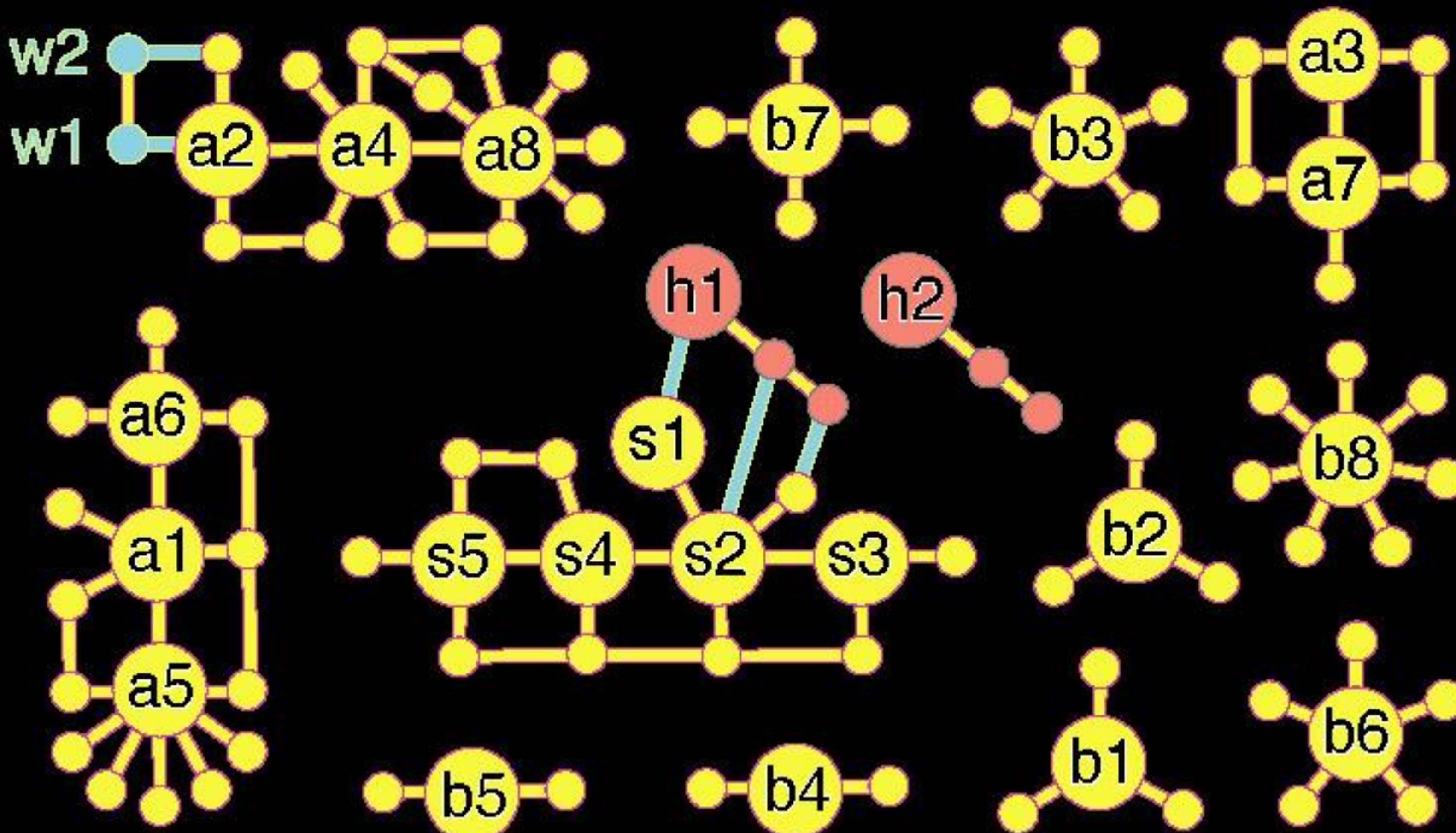
In all cases I have examined, $\dim H^1(\Delta,\mathbb{F}_2) \leq 4$.

For any given plane Π,

- compute $G = \mathrm{Aut}(\Pi)$;
- find a representative τ for each conjugacy class of involutions in G;
- compute H^1;
- if $H^1 \neq 0$, find orbits of $\mathrm{Aut}(\Pi/\tau)$ on H^1.

Known Planes of Order 25



Translation planes a1,…,a8; b1,…,b8; s1,…,s5 classified by Czerwinski & Oakden (1992)

Other instances of non-unique lifting (among planes of order 16)

Johnson-Walker plane

Dempwolff plane

quotient

Other instances of non-unique lifting (among planes of order 16)

Lorimer-Rahilly plane

derived semifield plane

quotient

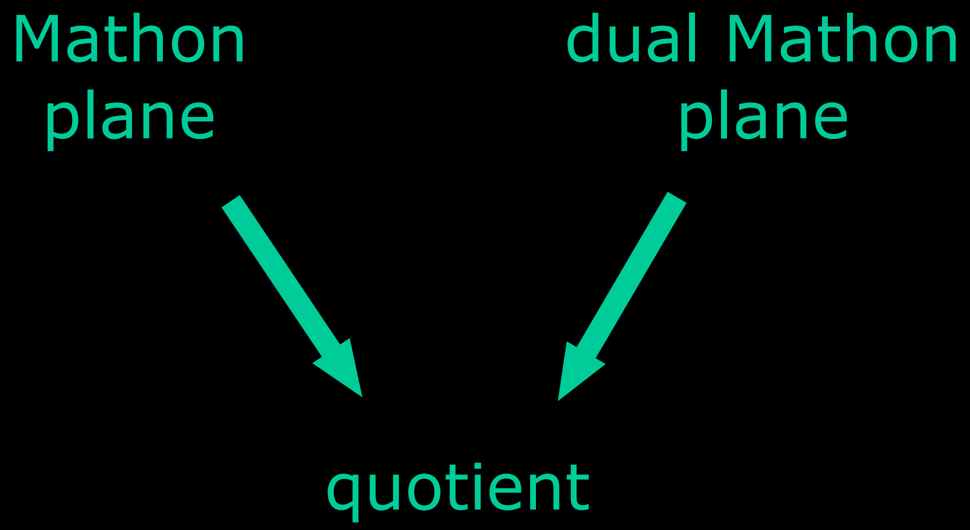
Other instances of non-unique lifting (among planes of order 16)

semifield plane over $\mathbb{F}_4$

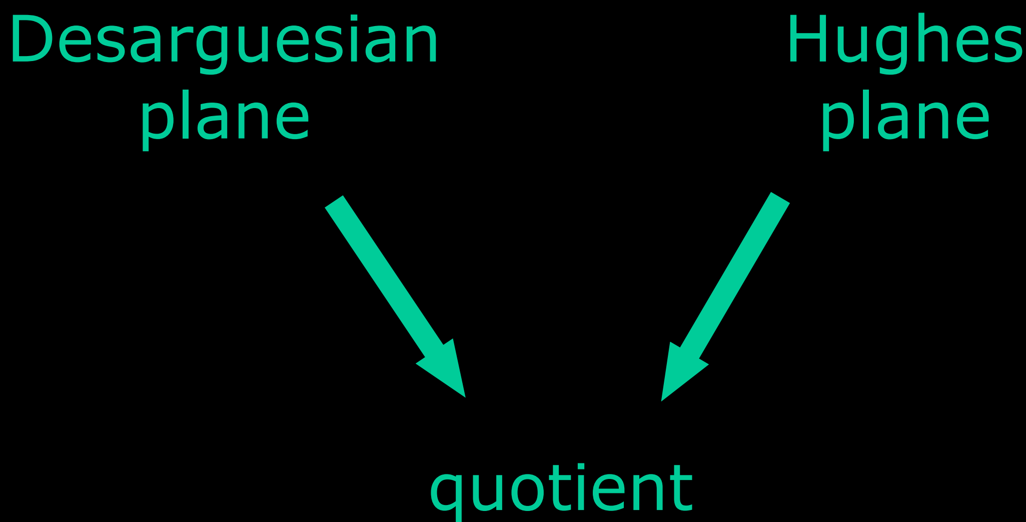
semifield plane over $\mathbb{F}_2$

quotient

Other instances of non-unique lifting (among planes of order 16)



Other instances of non-unique lifting (among planes of order 9)



Why only consider *involutions* $\tau \in \mathrm{Aut}(\Pi)$?

In this case the problem of lifting Π/τ to Π amounts to solving a linear system.

In any double cover, the fibres are necessarily τ-orbits for some involution τ.

Also tested:

- the 18 smallest known generalised quadrangles;
- the 4 smallest known generalised hexagons;
- the smallest known generalised octagon.

In only one case is H^1 nontrivial:
$\dim H^1 = 1$ for the GQ with $s=3$, $t=5$.

We are having to solve linear systems over $\mathbb{F}_2$ with thousands of unknowns.

Space constraints (computer memory) is the chief limitation in testing larger generalised polygons.