Automorphism Groups of Projective Planes with Arbitrarily Many Point and Line Orbits

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joint work with Tim Penttila

Projective Planes

A *projective plane* is a point-line incidence structure for which

- every pair of distinct points lies on a unique line;
- every pair of distinct lines meets in a unique point; and
- there exist four points with no three collinear.

Every point lies on $N+1$ lines, and every line has $N+1$ points, where N is the *order* of the plane (finite or infinite).

There are N^2+N+1 points and the same number of lines. In the infinite case, this number is simply N.

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Dembowski-Hughes-Parker Theorem

a.k.a. Block's Lemma

Theorem (c. 1950's)

Let G be an automorphism group of a finite projective plane Π. Then G has equally many point and line orbits.

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Proof (Brauer, 1941).

Let Π be a finite projective plane with incidence matrix A, and let G be an automorphism group of Π. We have two permutation representations $\pi_i : G \mapsto GL_{N^2+N+1}(\mathbb{C})$ satisfying

$$\pi_1(g)^{-1} A \pi_2(g) = A \quad \text{for all } g \in G.$$

Here π_1, π_2 are the actions of G on points and lines respectively. Now

$$\pi_2(g) = A^{-1} \pi_1(g) A \quad \text{for all } g \in G$$

so

$$[\chi_1, 1_G] = [\chi_2, 1_G]$$

where $\chi_i(g) = tr\, \pi_i(g)$, i.e. the number of G-orbits on points equals the number of G-orbits on lines. □

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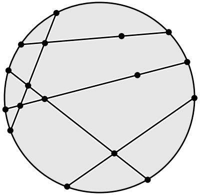
A *Near*-Example

Cameron mentions the following infinite design which comes close to what is required: Start with a closed disk D. Consider the 2-design

$\mathfrak{D}$: Points of D and Lines=Chords of D



Aut $\mathfrak{D}$ is transitive on lines (i.e. chords). It has two orbits on points (boundary points and interior points).

But $\mathfrak{D}$ is not a projective plane: two chords meet in 0 or 1 points.

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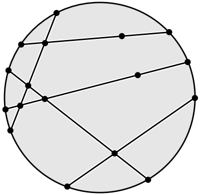

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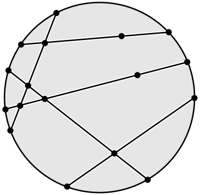

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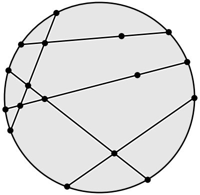

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Theorem (M. and Penttila, 2012)

- *There exists a Desarguesian plane* Π *admitting a group* $G < \mathrm{Aut}\,\Pi$ *having two orbits on points, and more than two orbits on lines.*
- *Given any two nonempty sets* A *and* B*, there exists a projective plane* Π *admitting a group* $G \leqslant \mathrm{Aut}\,\Pi$ *having exactly* $|A|$ *orbits on points and* $|B|$ *orbits on lines.*

Skewfields
Artin's Problem

Consider an extension of skewfields $L \supseteq K$.

Artin (1946) asked whether it is possible for the left and right degrees of L over K to differ.

Cohn (1961) gave examples with one degree infinite and the other degree an arbitrary integer $n \geqslant 2$. Schofield (1985) gave examples where the left and right degrees are arbitrary integers $m, n \geqslant 2$.

For our construction, take left-degree > 2 and right-degree 2.

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Desarguesian planes

An arbitrary Desarguesian plane $\Pi = PG(2, L)$ is coordinatized by a (possibly commutative) skewfield L. Points and lines correspond to left and right L-subspaces of L^3 of dimension 1, respectively:

Typical point: $P = L(a,\ b,\ c) = \{(\lambda a,\ \lambda b,\ \lambda c) : \lambda \in L\} \neq \{(0,0,0)\}$

$$\text{Typical line:}\quad \ell = \begin{pmatrix} d \\ e \\ f \end{pmatrix} L = \left\{ \begin{pmatrix} d\mu \\ e\mu \\ f\mu \end{pmatrix} : \mu \in L \right\} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Incidence:}\quad P \in \ell \ \Leftrightarrow\ (a,\ b,\ c) \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf = 0$$

$\mathrm{Aut}(\Pi) \cong PGL_3(L) = GL_3(L)/Z$ where $Z = \{\lambda I : 0 \neq \lambda \in Z(L)\}$

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Desarguesian examples

Consider $G = PGL_3(K) < PGL_3(L)$ where $L \supseteq K$ has left-degree > 2 and right-degree 2.

Then G has 2 orbits on points of $\Pi = PG_2(L)$:

$$L(a,\ b,\ c) \in \begin{cases} (L(1,\ 0,\ 0))^G, & \text{if } a, b, c \text{ are right-linearly dependent over } K; \\ (L(1,\ \alpha,\ 0))^G, & \text{if } a, b, c \text{ are right-linearly independent over } K \end{cases}$$

where $\{1, \alpha\}$ is a basis for L as a right vector space over K. And *more than* 2 orbits on lines.

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The Group G

Consider a multiplicative group G satisfying

(G1) G is infinite nonabelian;

(G2) Every conjugacy class in G other than $\{1\}$ has cardinality $|G|$; and

(G3) Every element of G has at most one square root in G.

For every infinite cardinal number C, there is such a group of cardinality C (e.g. a free group on C generators; in the countable case, 2 generators suffice).

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Plane Construction

Theorem

Let A and B be nonempty sets with $|A|, |B| \leqslant |G|$ where G satisfies (G1), (G2), (G3) above. Then there exists a projective plane Π of order $|G|$ with a group of collineations isomorphic to G, having exactly $|A|$ point orbits and $|B|$ line orbits.

Proof We require an indexed collection of subsets $D_{a,b} \subset G$ for $(a, b) \in A \times B$ satisfying certain conditions (see (D1), (D2) below).

$$\text{Points:} \quad (x, a) \in G \times A$$

$$\text{Lines:} \quad (y, b) \in G \times B$$

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The properties required of the subsets $D_{a,b} \subset G$ are:

(D1) For all $b_1, b_2 \in B$ and $g \in G$, there exists $a \in A$ and elements $d_i \in D_{a,b_i}$ such that $g = d_1^{-1}d_2$. The triple (a, d_1, d_2) is unique whenever $(b_1, g) \neq (b_2, 1)$.

(D2) For all $a_1, a_2 \in A$ and $g \in G$, there exists $b \in B$ and elements $d_i \in D_{a_i,b}$ such that $g = d_1 d_2^{-1}$. The triple (b, d_1, d_2) is unique whenever $(a_1, g) \neq (a_2, 1)$.

We construct the required subsets $D_{a,b} \subset G$ by transfinite recursion. When $|A| = |B| = 1$ just one subset $D \subset G$ is required, a *difference set* (Hughes, 1955).

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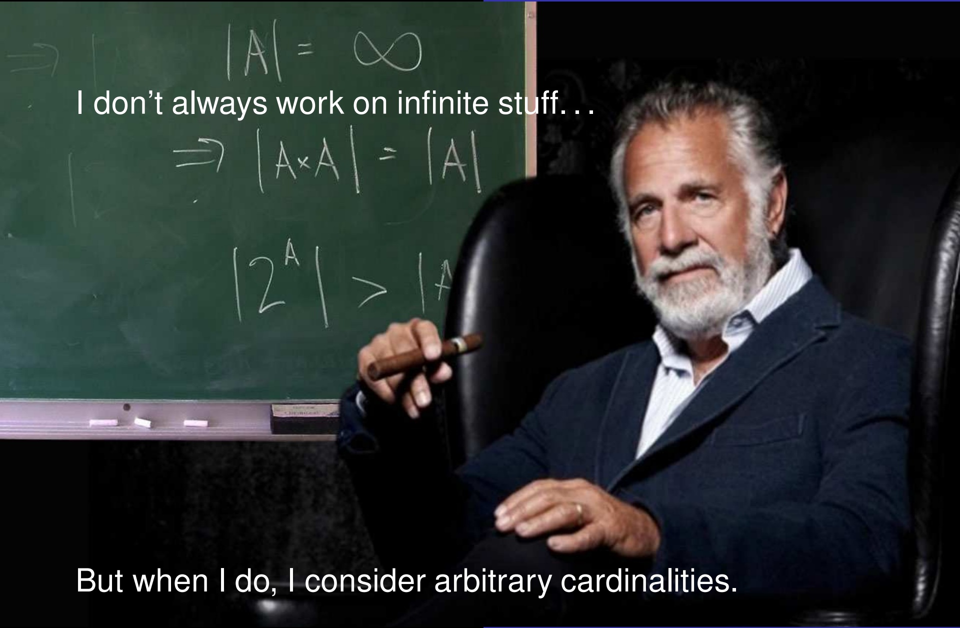
I don't always work on infinite stuff. . .
$|A| = \infty$
$\Rightarrow |A \times A| = |A|$
But when I do, I consider arbitrary cardinalities.

Recursion on STEPS $= (A \times A \times G) \cup (B \times B \times G)$

Let

$$\text{STEPS} = (A \times A \times G) \cup (B \times B \times G)$$

(here we assume $A \cap B = \varnothing$). Well-order the set of steps as

$$\text{STEPS} = \{\text{STEP}(\alpha) : \alpha < C\}$$

where $C = |\text{STEPS}| = |G|$. Recursively construct

$$D_{a,b} = \bigcup_{\alpha < C} D_{a,b}(\alpha).$$

Initially (i.e. at STEP(0)) all sets $D_{a,b}(0) = \varnothing$.

For every limit ordinal $\alpha < C$, set $D_{a,b}(\alpha) = \bigcup_{\beta<\alpha} D_{a,b}(\beta)$.

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1. Suppose $g = d_1^{-1} d_2,\ d_i \in D_{a,b_i}(\beta),\ a \in A$. Then *add nothing*: $D_{a,b}(\alpha) = D_{a,b}(\beta)$ for all a, b.
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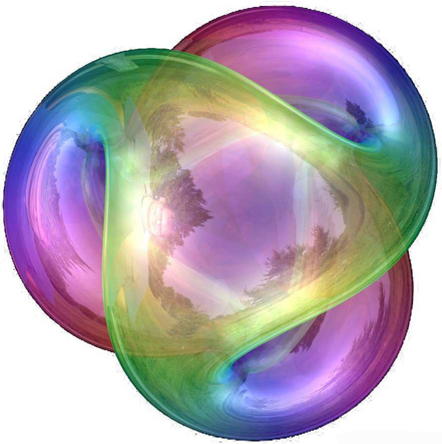
2. If 1 fails, first choose $a_1 \in A$ arbitrarily. Form $D_{a,b}(\alpha) \supseteq D_{a,b}(\beta)$ by adding one or two new elements for $(a,b) \in \{(a_1,b_1),(a_1,b_2)\}$ and no new elements for other (a,b), such that $g = d_1^{-1}d_2,\ d_i \in D_{a_1,b_i}(\alpha)$. There are $|G| = C$ elements to choose from, and *fewer* than this many choices are excluded by (D1)–(D2).

Thank You!



Questions?