Partial Spreads and Flocks over Infinite Fields

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Spreads

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A *spread* is a set of mutually skew lines which partitions the points (i.e. deficiency=0).

A *dual spread* is a set Σ of mutually skew lines such that every plane contains a unique $\ell \in \Sigma$. (For $|F| < \infty$, a spread is the same as a dual spread.)

A *bispread* is a spread which is also a dual spread.

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If $\mathcal{B} = \{B_1, \ldots, B_n\}$ *are distinct points of* $\mathbb{P}^3\mathbb{Q}$*, then there exists a partial spread of* $\mathbb{P}^3\mathbb{Q}$ *of deficiency n omitting precisely the points of* $\mathcal{B}$*.*

Proof. Enumerate the points of $\mathbb{P}^3\mathbb{Q} \smallsetminus \mathcal{B}$ as $P_0, P_1, P_2, \ldots$. Inductively define a chain of partial spreads

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

such that $P_i \in \bigcup \Sigma_j$ whenever $i < j$; and the partial spread $\Sigma = \bigcup_{j \geqslant 0} \Sigma_j$ satisfies $\bigcup \Sigma = \mathbb{P}^3\mathbb{Q} \smallsetminus \mathcal{B}$. Define $\Sigma_0 = \varnothing$ and

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Improvement, also by induction:

Theorem

Every finite partial spread of $\mathbb{P}^3\mathbb{Q}$ can be completed to a spread.

Moreover if $\Sigma = \{\ell_1, \ldots, \ell_m\}$ is a partial spread and $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$ are distinct points not in $\bigcup\Sigma$, then Σ may be extended to a partial spread of deficiency n omitting precisely the points of $\mathcal{B}$.

A similar result holds for *real* projective 3-space, by transfinite induction:

Theorem

Every partial spread Σ of $\mathbb{P}^3\mathbb{R}$ of cardinality $|\Sigma| < 2^{\aleph_0}$ can be completed to a spread.

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In particular $\mathbb{R}^3 \setminus \{O\}$ can be partitioned into lines. Is there a direct way to show this (without transfinite induction)?

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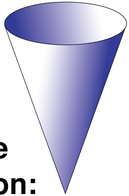
Quadratic Cones

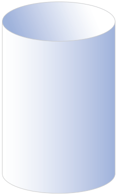
Projective description:



(W, X, Y, Z) such that

$$\underbrace{\alpha X^2 + \beta XY + \gamma Y^2}_{\textit{irreducible}} = W^2$$

Affine description:



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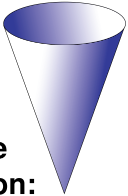

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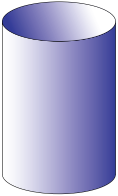

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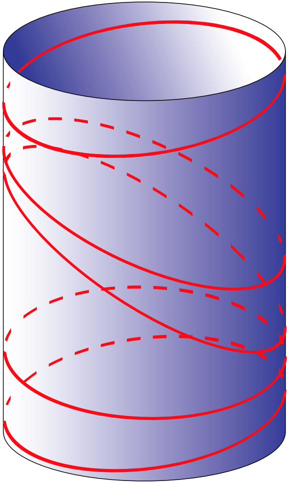
A **partial flock of the cone** is a collection of mutually disjoint conics (plane sections) of the cylinder (i.e. the cone minus its vertex).

The **deficiency** of a partial flock is the number of points of the cylinder not covered.

A **flock of the cone** is a partition of the points of the cylinder into conics (deficiency=0).



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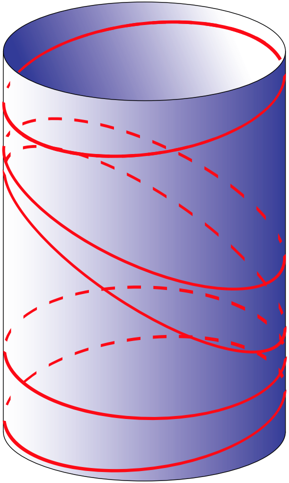

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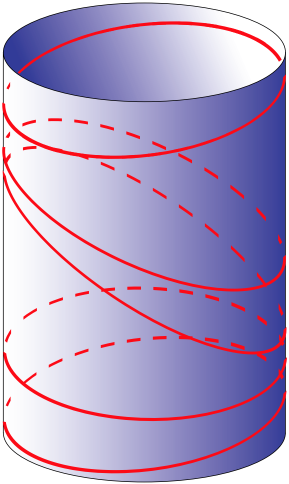

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$\mathbb{F}_q$	$k(q+1)$	$2+k(q+1)$	$k(q+1)$	$0 \leqslant k \leqslant q+\varepsilon$ *
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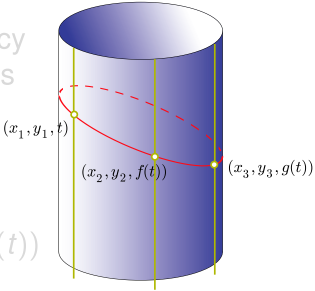
A partial flock of a real quadratic cone has deficiency 0 or $2^{\aleph_0}$.

Proof. Consider a partial flock of deficiency $< 2^{\aleph_0}$. Then there exist at least three lines of the cylinder, each of which is covered by the flock. Ellipses of the flock meet these lines in points



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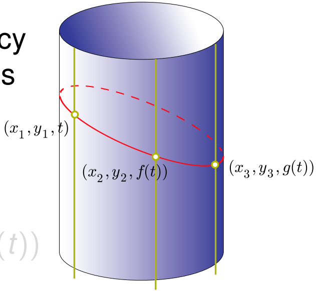

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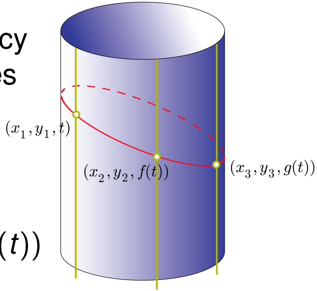

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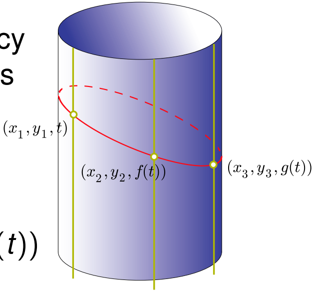

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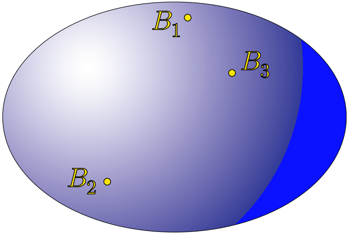

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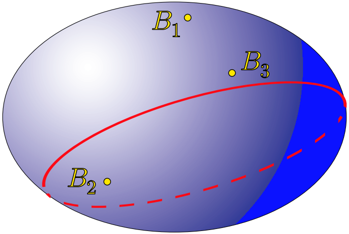

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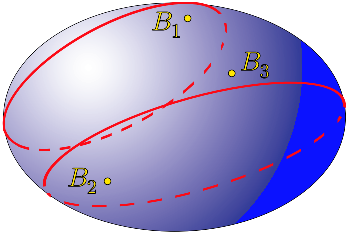

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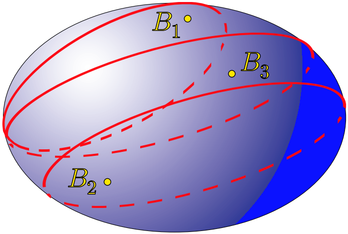

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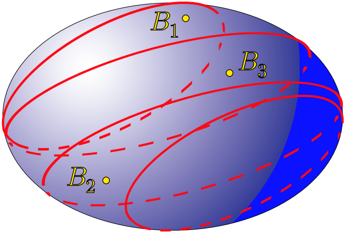

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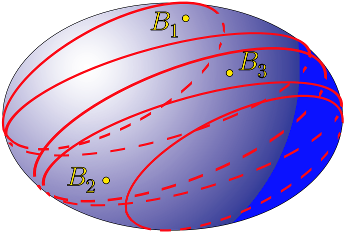

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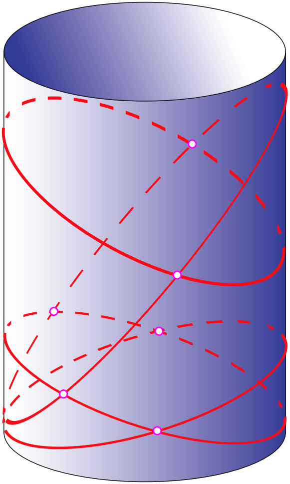
We may replace $\mathbb{Q}$ by any countable subfield of $\mathbb{R}$.



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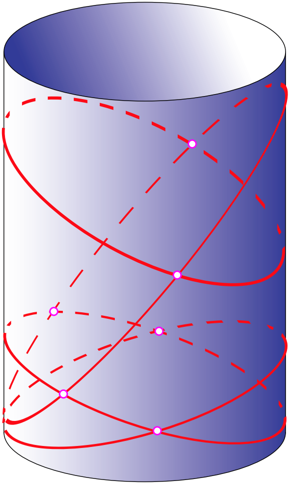

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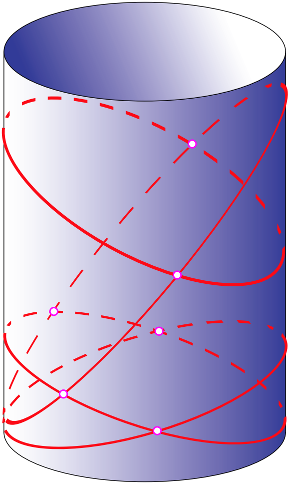

Shown: a typical partial flock of a cone over $\mathbb{Q}$

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The p-adic norm

Let p be a prime. Define the p-adic norm of a rational number by

$$\|p^r \tfrac{a}{b}\|_p = p^{-r}$$

where $a, b, r \in \mathbb{Z}$ with $p \not| \, ab$; also

$$\|0\|_p = 0.$$

This satisfies

(i) $\|x+y\|_p \leqslant \max\{\|x\|_p, \|y\|_p\}$ with equality whenever $\|x\|_p \neq \|y\|_p$; and

(ii) $\|xy\|_p = \|x\|_p \|y\|_p$.

For every *nonzero* $x \in \mathbb{Q}$,

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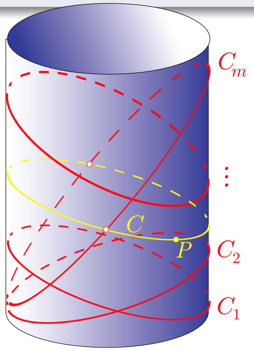
Lemma

Let $C_1, \ldots, C_m$ be ellipses on the affine cylinder $X^2 + Y^2 = 1$ over $\mathbb{Q}$, and let $P \notin C_1 \cup C_2 \cup \cdots \cup C_m$ be a point of the cylinder. Then there exists an ellipse C on the cylinder containing P, and disjoint from each of the ellipses $C_1, C_2, \ldots, C_m$.



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Proof. WLOG $P = (1, 0, 0)$ since the cylinder admits a point-transitive group of automorphisms

$$(x, y, z) \mapsto (ax+by, -bx+ay, z+c),\ a^2+b^2=1,\ a, b, c \in \mathbb{Q}.$$

$$C_i\ :\ Z=\alpha_i X+\beta_i Y+\gamma_i,\ i=1,2,\ldots,m;\ \alpha_i, \beta_i, \gamma_i \in \mathbb{Q};\ \alpha_i+\gamma_i \neq 0.$$

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Choose distinct primes $p_1, \ldots, p_m \equiv 3 \mod 4$ such that $\|\alpha_i + \gamma_i\|_{p_i} = 1$.

By Weak Approximation, there exist $\alpha, \beta \in \mathbb{Q}$ such that

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for $i = 1, 2, \ldots, m$. The planes of $\mathcal{C}$ and $\mathcal{C}_i$ intersect in a line which satisfies

$$(\alpha - \alpha_i)X + (\beta - \beta_i)Y = \alpha + \gamma_i$$

where

$$\|\alpha + \gamma_i\|_{p_i} = \|(\alpha - \alpha_i) + (\alpha_i + \gamma_i)\|_{p_i} = 1.$$

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We may *also* prescribe points $B_1, \ldots, B_n$ of the cylinder, and insist that $\mathcal{C}$ does not contain any of these.

Theorem

Let $B_1, \ldots, B_n$ be distinct rational points on the cylinder $X^2 + Y^2 = 1$. Let $\mathcal{C}$ be a partial flock of the cylinder not covering any of the points B_i. Then $\mathcal{C}$ extends to a partial flock of the cylinder covering all points of the cylinder except $B_1, \ldots, B_n$.

Proof. Enumerate the rational points of the cylinder *other than* $B_1, \ldots, B_n$ as $P_0, P_1, P_2, \ldots$. Using the Lemma we inductively define a sequence of partial flocks

$$\mathcal{C}_0 = \mathcal{C} \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots$$

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Ovoids

Let $\mathcal{P}$ be a classical polar space (e.g. a nondegenerate quadric in $\mathbb{P}^n F$).

A *cap* in $\mathcal{P}$ is a set of points $\mathcal{C}$ in $\mathcal{P}$, such that every generator (i.e. maximal projective subspace) $U \subset \mathcal{P}$ satisfies $|U \cap \mathcal{C}| \leqslant 1$.

An *ovoid* is a set of points $\mathcal{O}$ in $\mathcal{P}$, such that every generator $U \subset \mathcal{P}$ satisfies $|U \cap \mathcal{O}| = 1$.

Theorem

Let $\mathcal{P}$ be a classical polar space over an infinite field F. If $\mathcal{C}$ is a cap in $\mathcal{P}$ with $|\mathcal{C}| < |F|$, then $\mathcal{C}$ extends to an ovoid of $\mathcal{P}$.

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