Approaching Some Problems in Finite Geometry Through Algebraic Geometry

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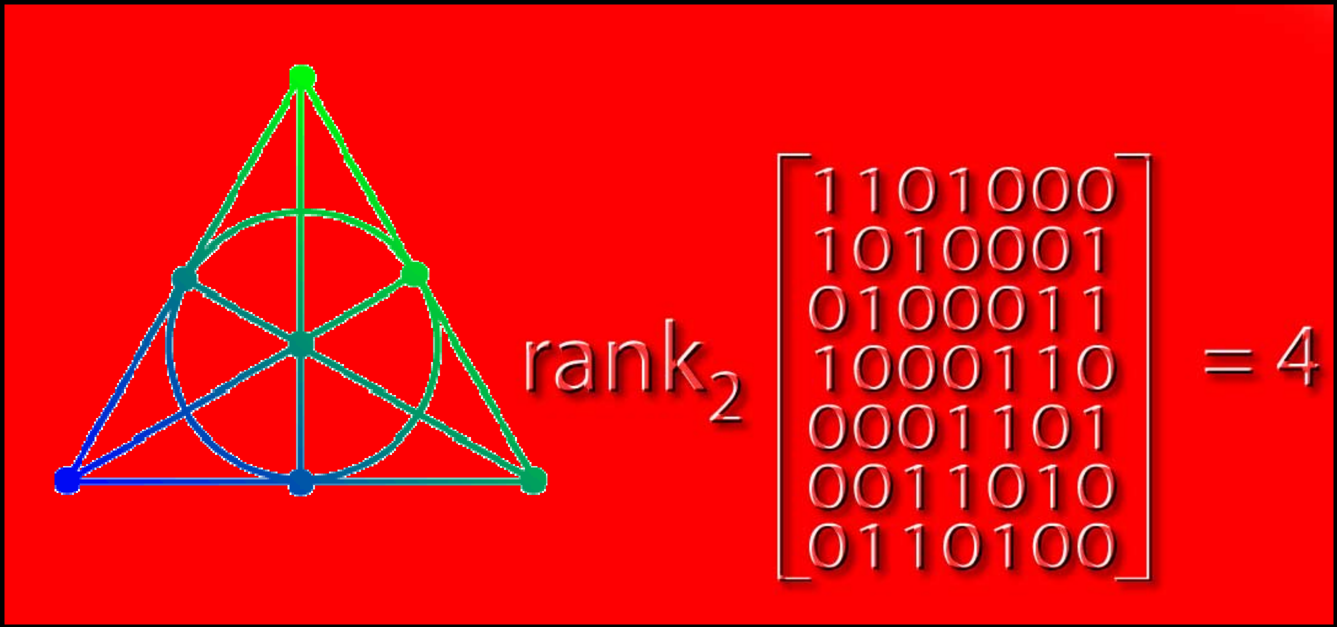
Algebraic Combinatorics

- finite geometry (classical and nonclassical)
- association schemes
- algebraic graph theory
- combinatorial designs
- enumerative combinatorics (à la Rota, Stanley, etc.)
- much more…

Use of Gröbner Bases: Conceptual vs. Computational

Outline

1. Motivation / Background from Finite Geometry
2. *p*-ranks
3. Computing *p*-ranks via the Hilbert Function
4. Open Problems



1. Motivation / Background from Finite Geometry

Classical projective n-space $P^n\mathbb{F}_q$:

incidence system formed by subspaces of $\mathbb{F}_q^{n+1}$

points = 1-spaces
lines = 2-spaces
planes = 3-spaces
etc.

Non-classical projective planes (2-spaces) exist but spaces of dimension ≥ 3 are classical

1. Motivation / Background from Finite Geometry

An *ovoid* in projective 3-space $P^3\mathbb{F}_q$:
a set $\mathcal{O}$ consisting of q^2+1 points, no three collinear.

Let $\mathcal{C}$ be a linear $[n,4]$ code over $\mathbb{F}_q$.
If $\mathcal{C}^\perp$ has minimum weight ≥ 4 then $n \leq q^2+1$.

When equality occurs then a generator matrix G for $\mathcal{C}$ has as its columns an ovoid.

1. Motivation / Background from Finite Geometry

An *ovoid* in projective 3-space $P^3\mathbb{F}_q$:
a set $\mathcal{O}$ consisting of q^2+1 points, no three collinear.

For q odd, an ovoid is an *elliptic quadric* [Barlotti (1955); Panella (1955)].

When q is even the *known* ovoids are the elliptic quadrics, and (when $q=2^{2e+1}$) the *Suzuki-Tits ovoids*.

1. Motivation / Background from Finite Geometry

A *spread* in projective $(2n-1)$-space $P^{2n-1}\mathbb{F}_q$:
a set $\mathcal{S}$ consisting of q^n+1 projective $(n-1)$-subspaces,
partitioning the points of $(2n-1)$-space.

These exist for all n and q, and give rise to *translation planes* (the most prolific source of non-classical projective planes).

1. Motivation / Background from Finite Geometry

Classical polar spaces of orthogonal, unitary, symplectic type :
projective subspaces of $P^n\mathbb{F}_q$ totally singular / isotropic with respect to the appropriate form, which induces a polarity

Orthogonal polar space: nondegenerate quadric
Unitary polar space: Hermitian variety

Projective and polar spaces constitute the
Lie incidence geometries of types A_n, B_n, C_n, D_n

1. Motivation / Background from Finite Geometry

Ovoid of a polar space $\mathcal{P}$:

a point set $\mathcal{O}$ meeting every maximal subspace of $\mathcal{P}$ exactly once

Spread of a polar space $\mathcal{P}$:

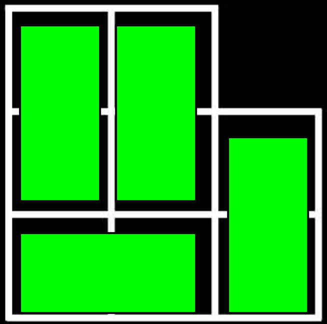
a partition $\mathcal{S}$ of the point set into maximal subspaces

Many existence questions for ovoids and spreads remain open.

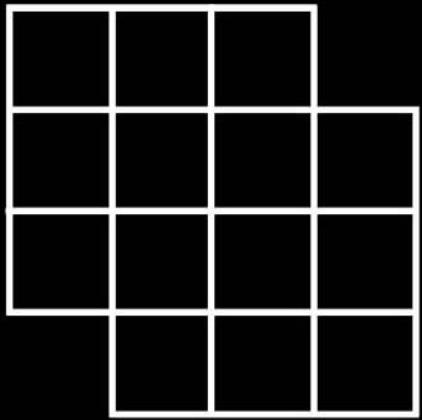
These may be regarded as dual packing problems:

Sample Packing Problem

Tile this figure
using 2×1 dominoes.



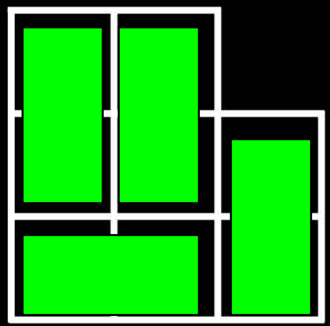
One, of several, solutions.
Such a complete tiling we'll call a *spread*.



This figure
has *no spread* of dominoes:

Sample Packing Problem

Tile this figure
using 2×1 dominoes.



One, of several, solutions.
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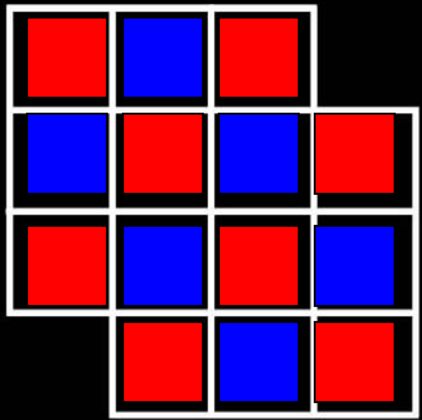
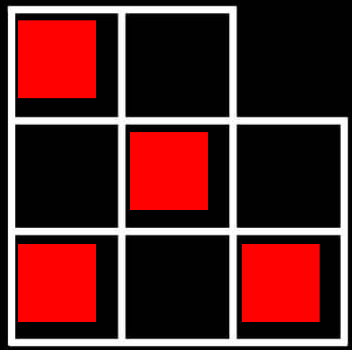
This figure
has *no spread* of dominoes:



The Dual Packing Problem

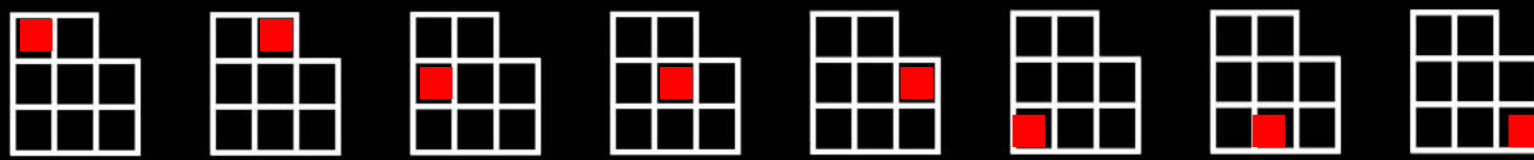
Find a set of cells meeting each domino exactly once.

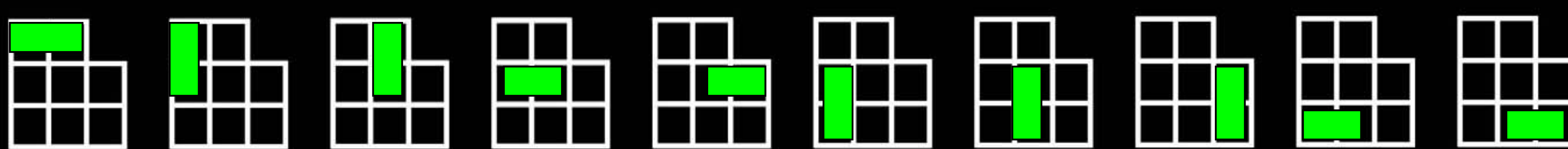


One of two solutions.
Such a set of cells we'll call an *ovoid*.

Why is this problem dual to the previous one?

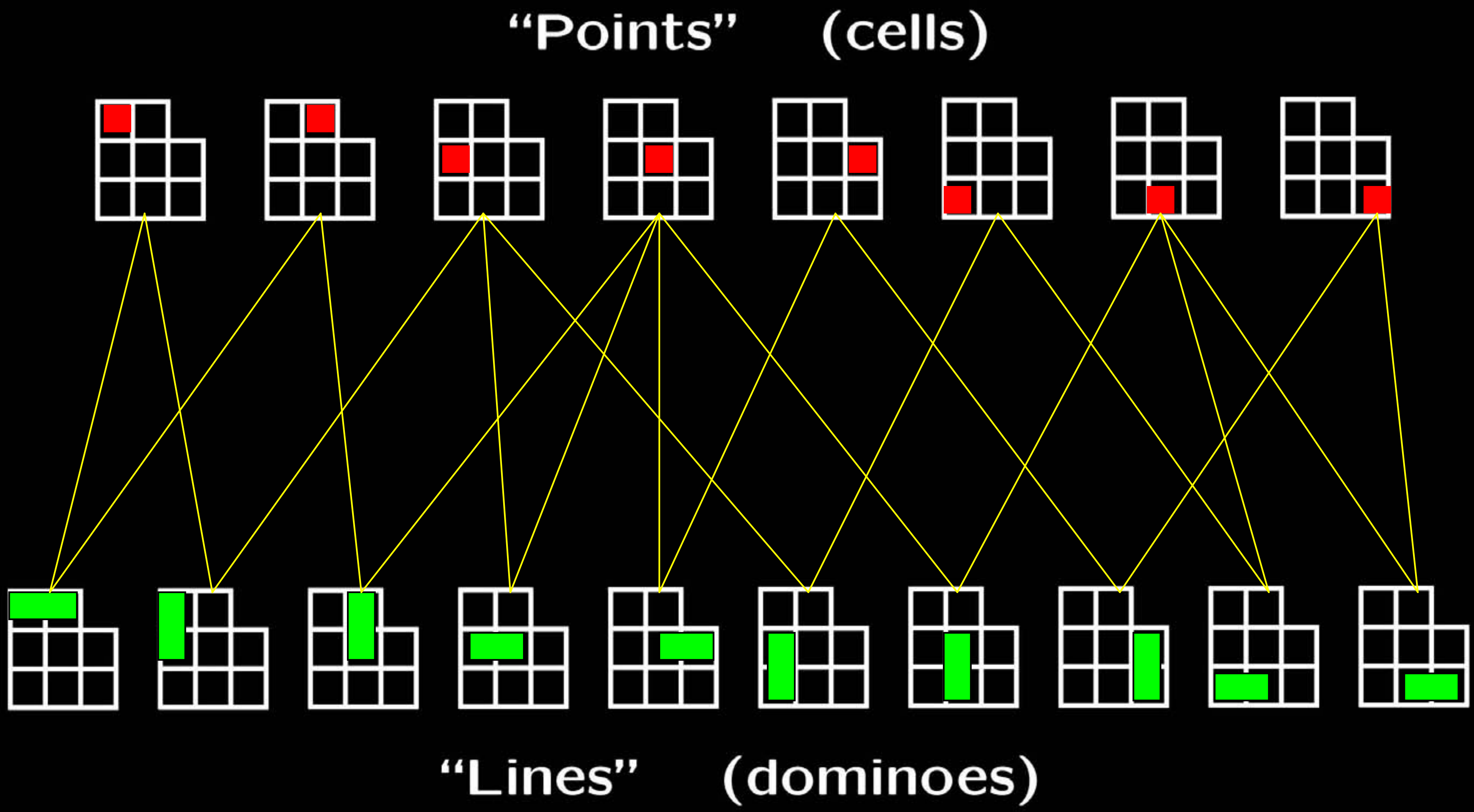
“Points” (cells)



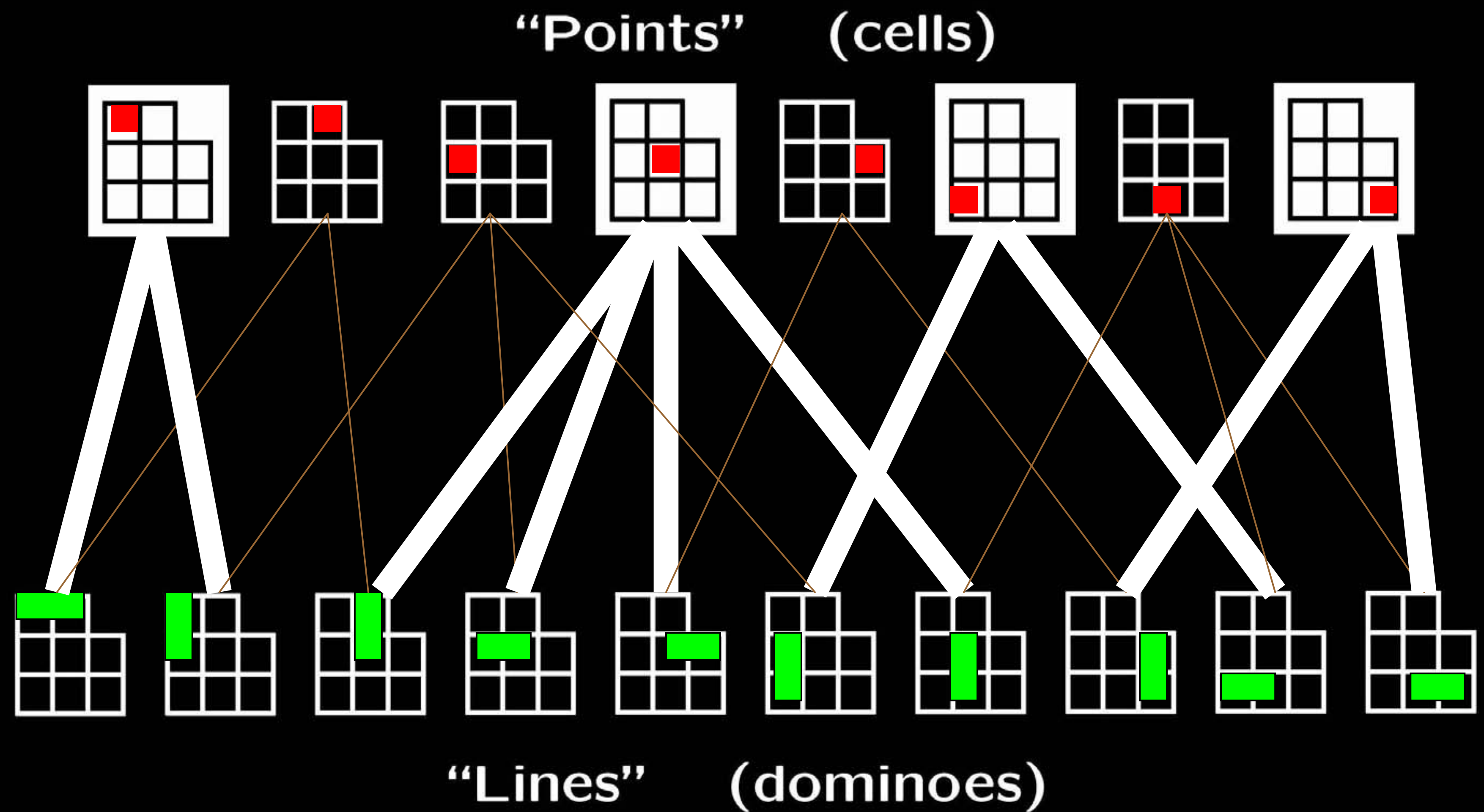


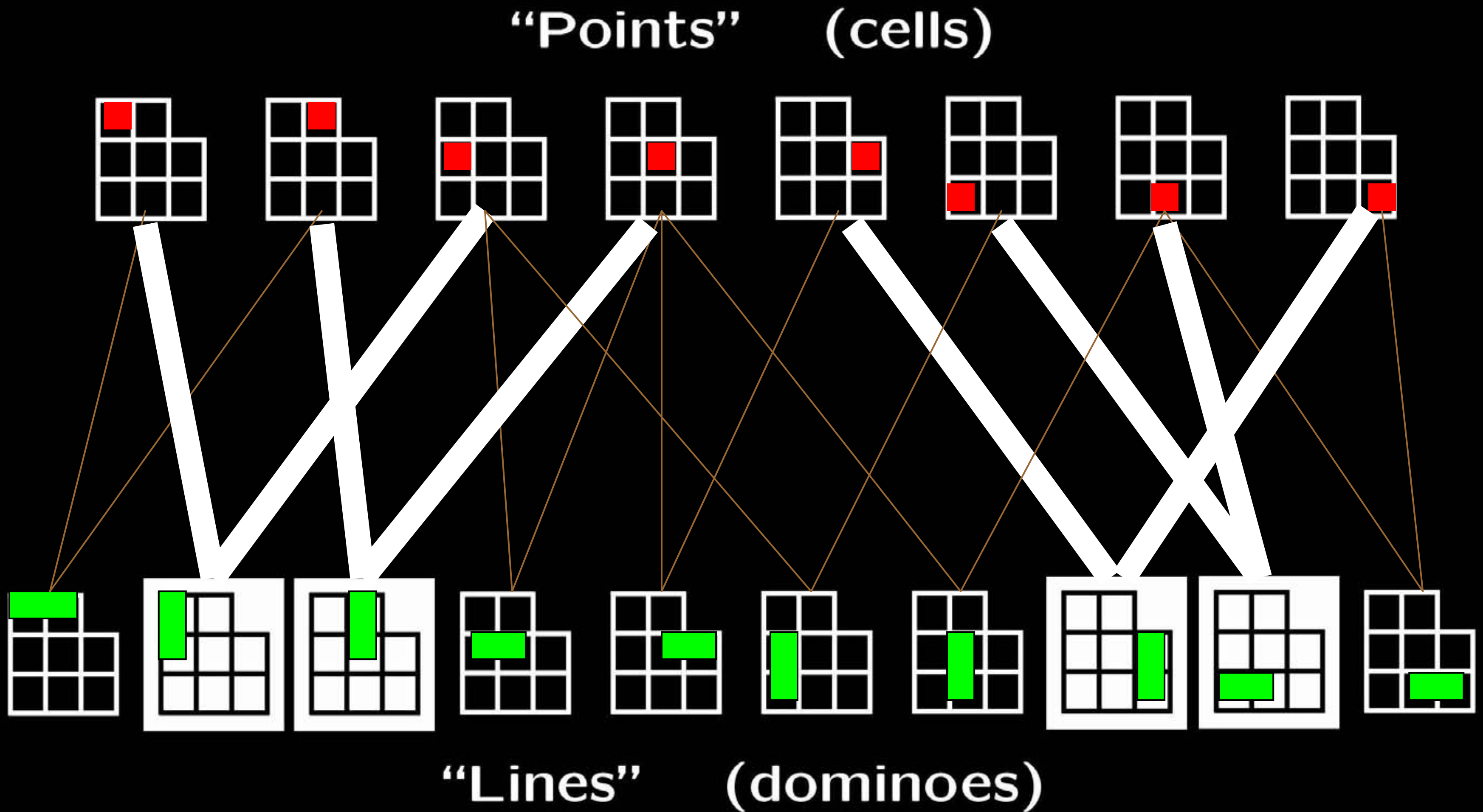
“Lines” (dominoes)

bipartite graph:



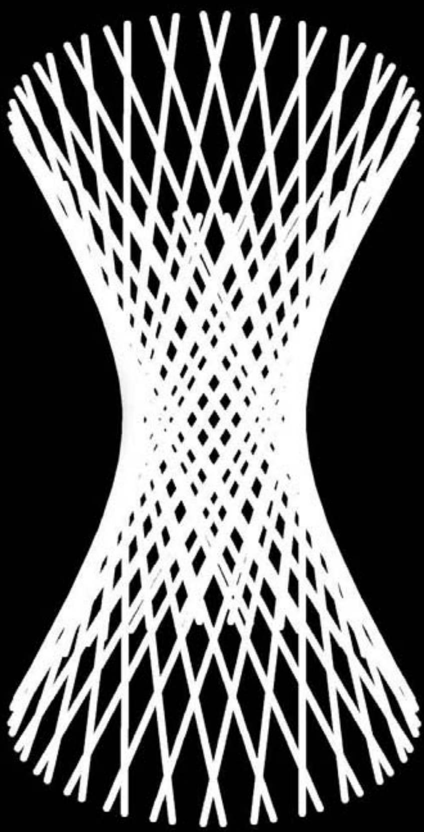
Ovoid



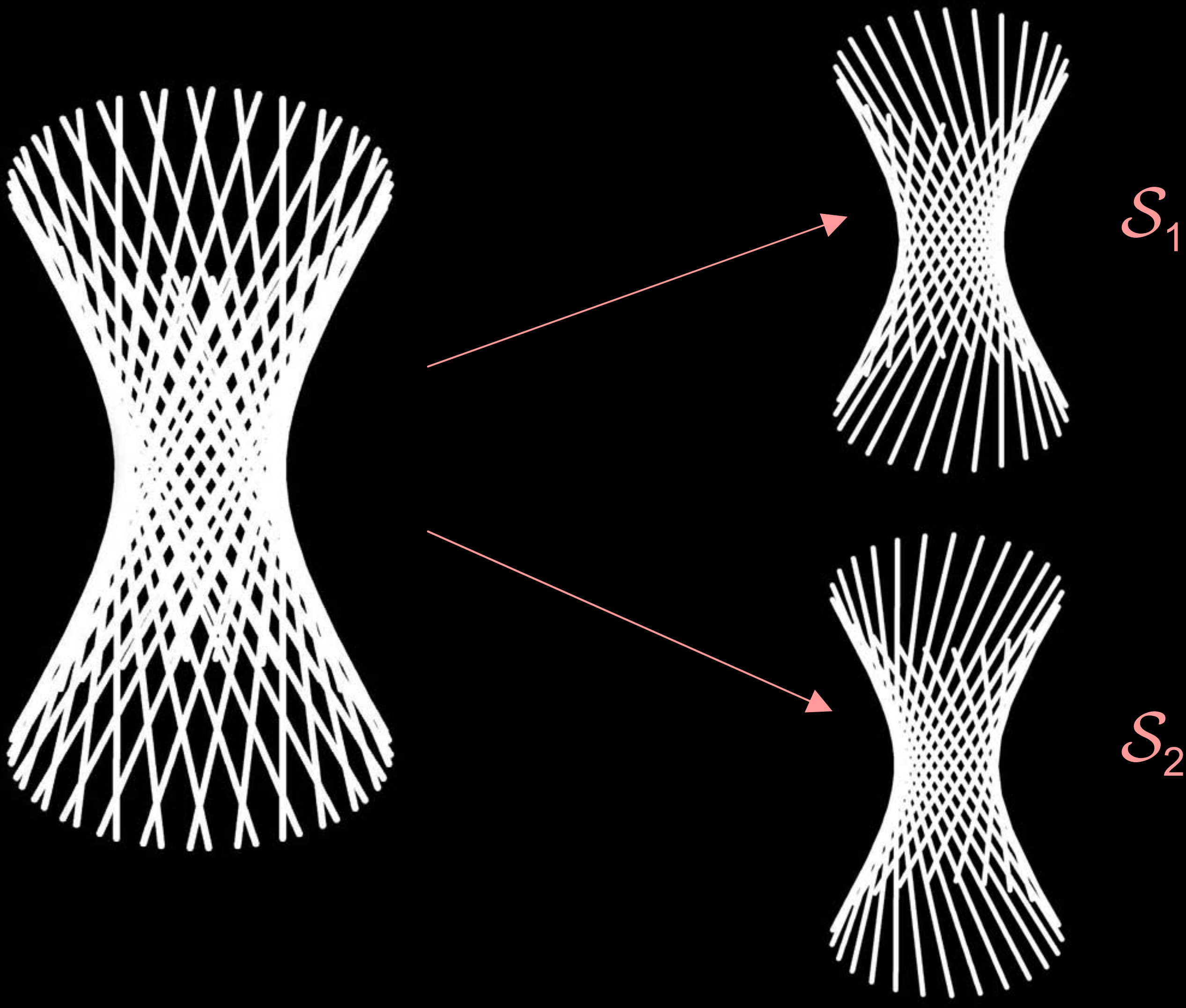


Spread

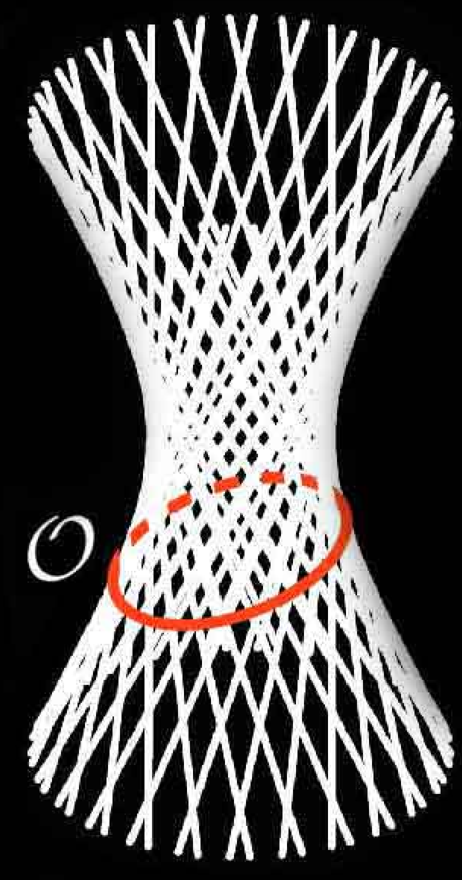
Hyperbolic (i.e. ruled) quadrics in P^3F



Hyperbolic (i.e. ruled) quadrics in P^3F have spreads

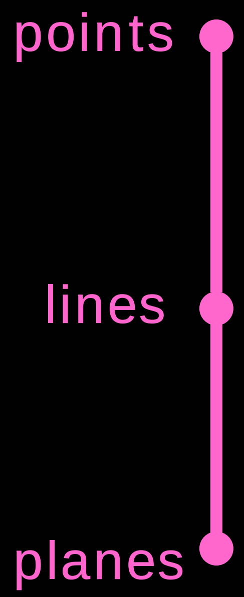


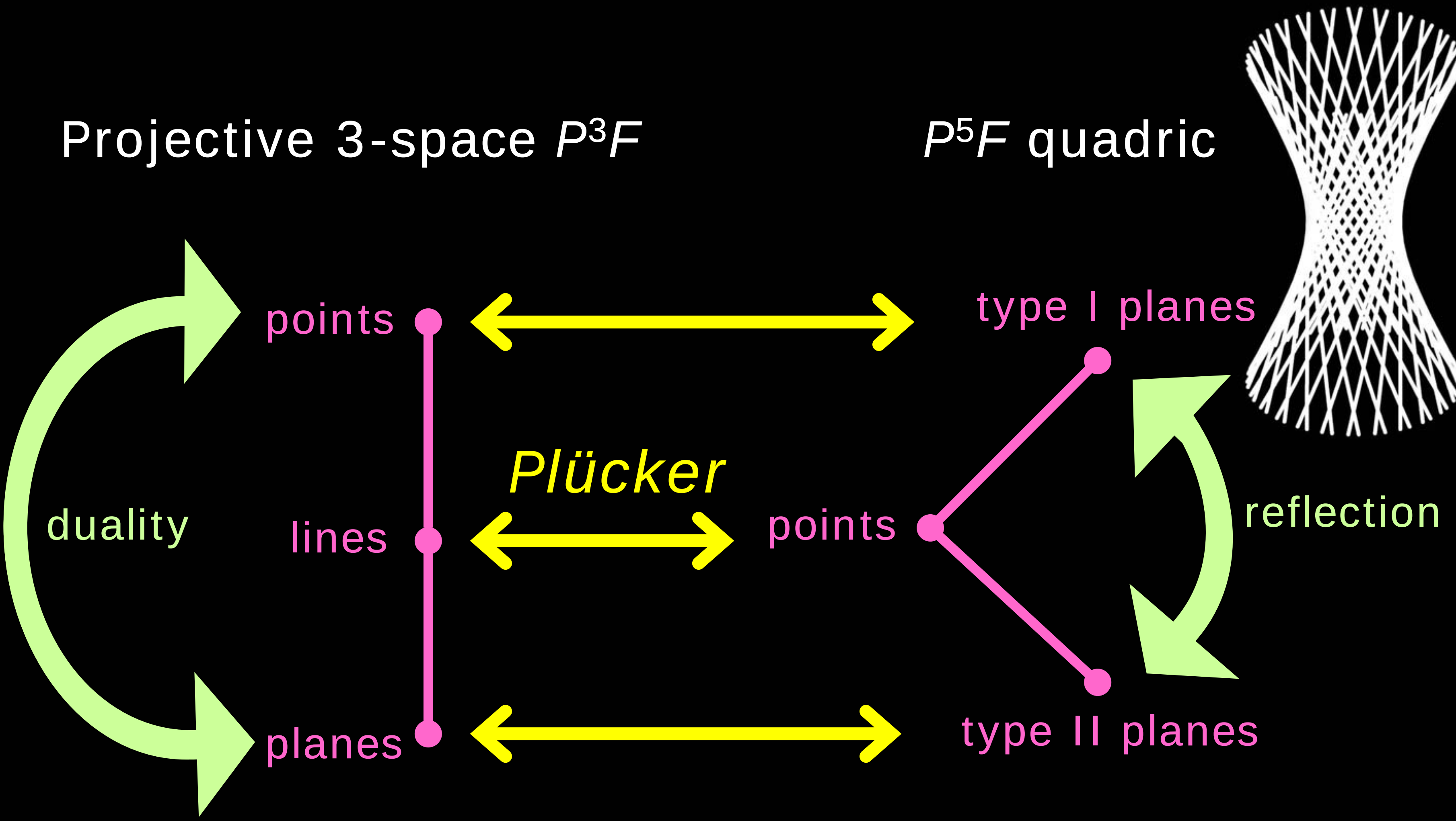
Hyperbolic (i.e. ruled) quadrics in P^3F have ovoids



All real quadrics have ovoids. Some have spreads.

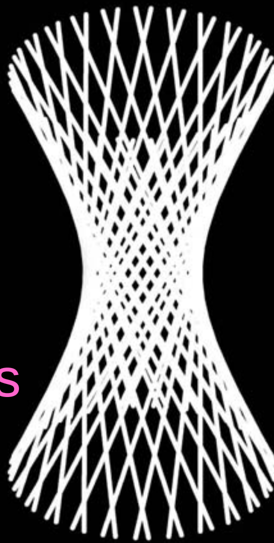
Projective 3-space P^3F

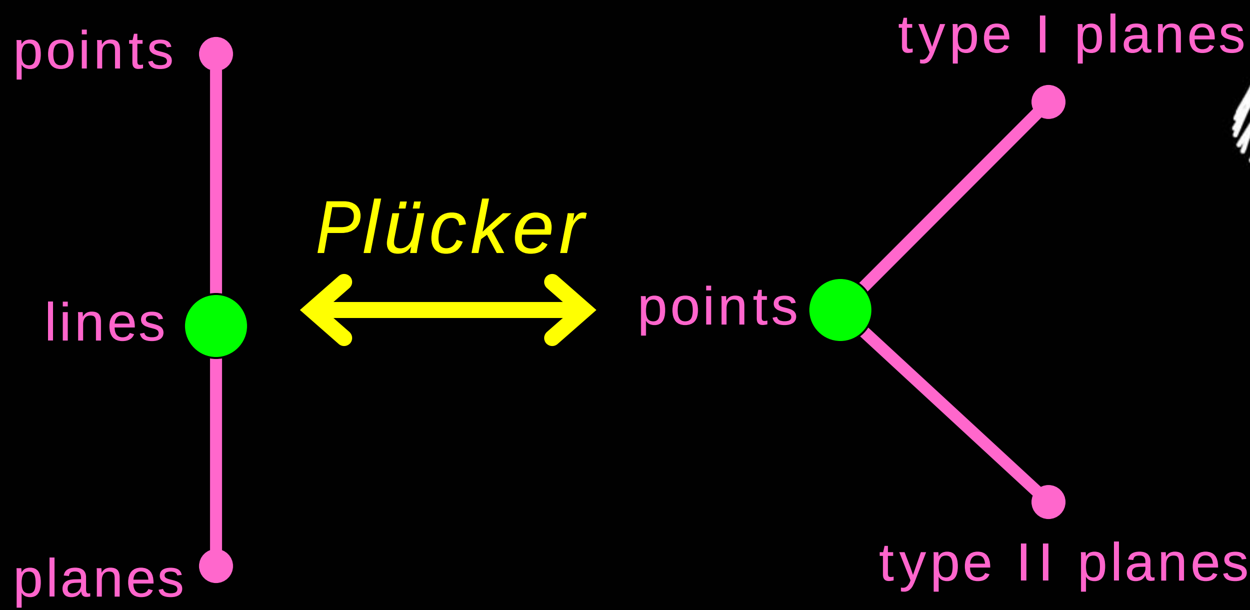



Projective 3-space P^3F
P^5F quadric
points
lines
planes
duality
Plücker
type I planes
points
type II planes
reflection

Projective 3-space P^3F

P^5F quadric





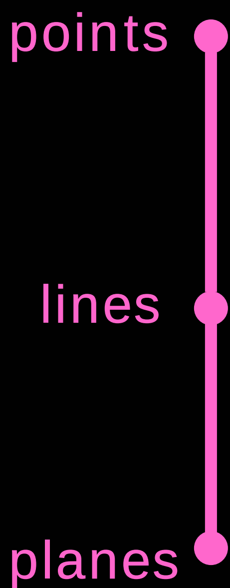
spread:

q^2+1 lines, pairwise disjoint

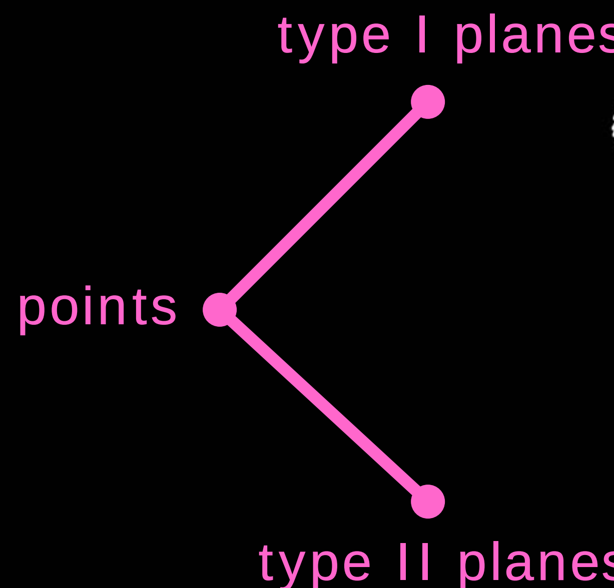
ovoid:

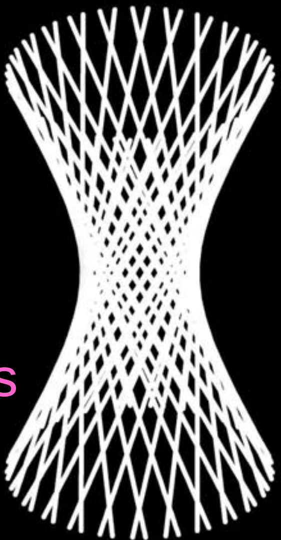
q^2+1 points, no two collinear

Projective 3-space P^3F



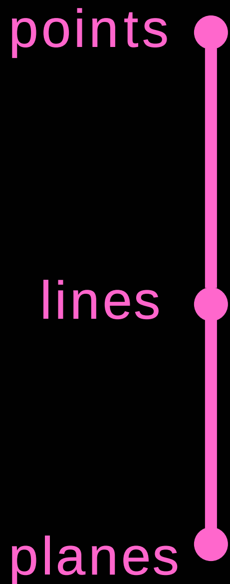
P^5F quadric

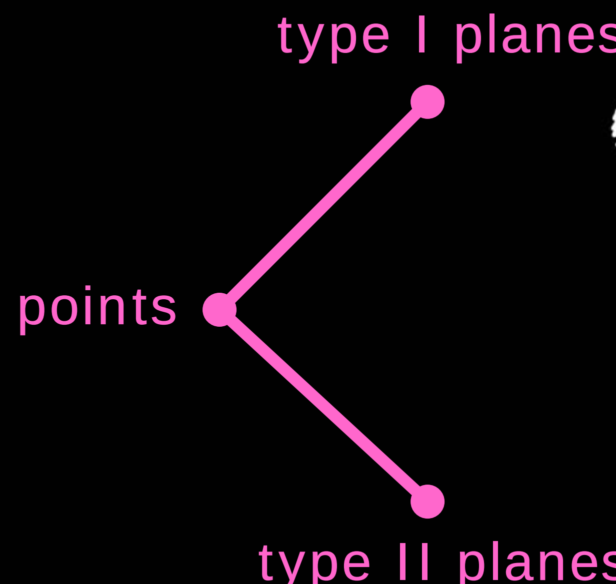




spread

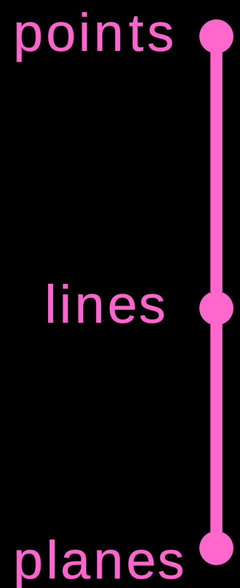
Projective 3-space P^3F | P^5F quadric





q^2+1 points (or planes), no two collinear ⟷ *Plücker* ⟷ spread

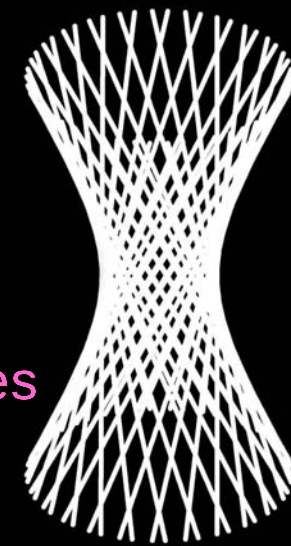
Projective 3-space P^3F

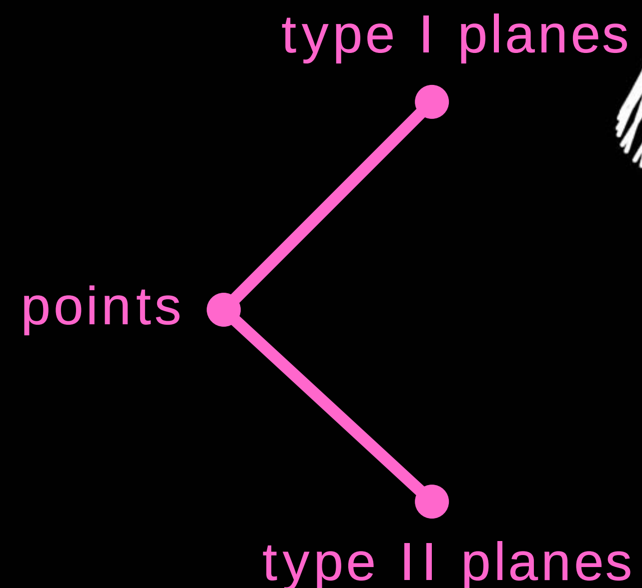


q^2+1 points (or planes), no two collinear

Plücker

P^5F quadric

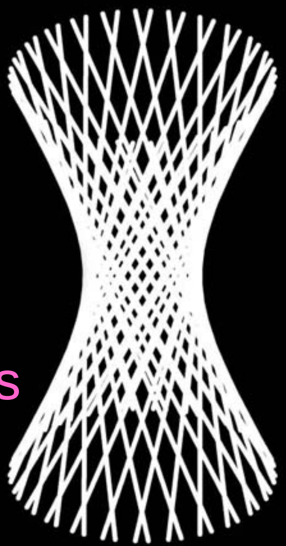


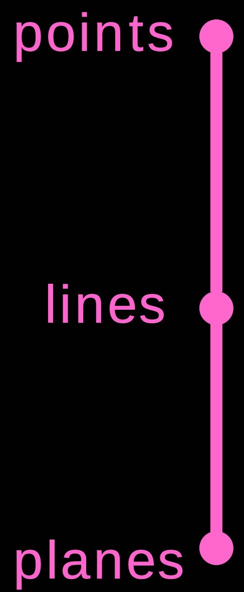


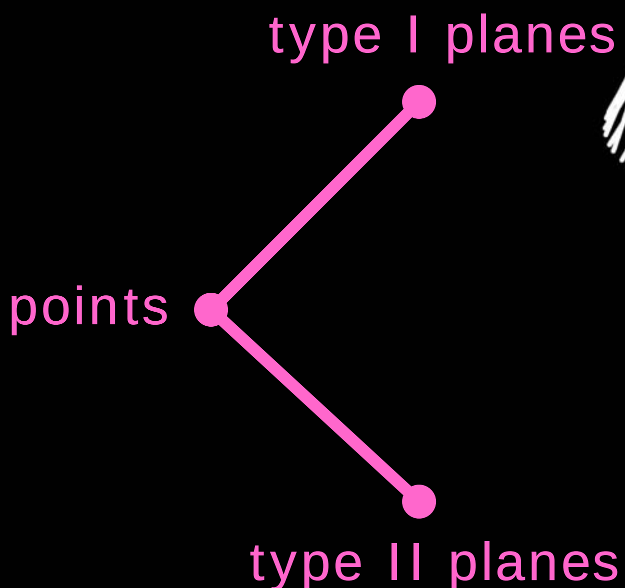
spread

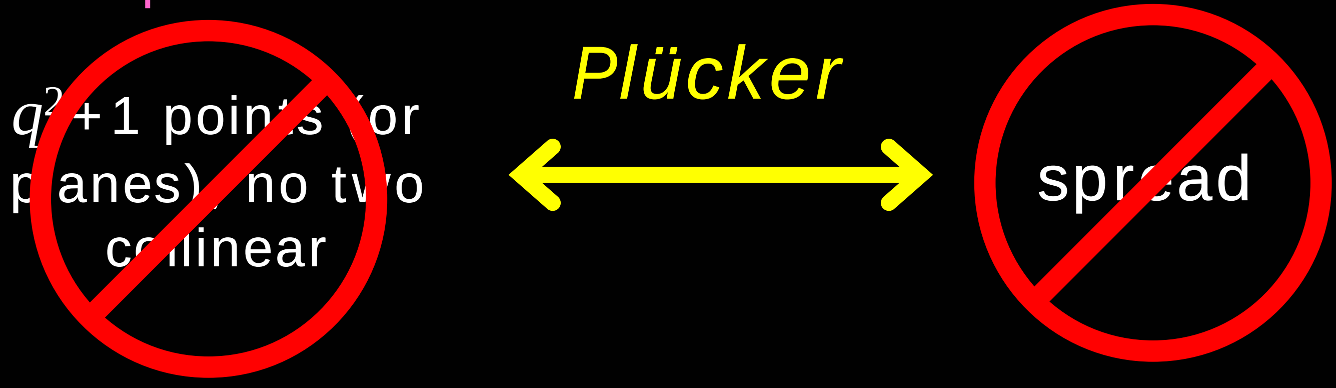
Projective 3-space P^3F

P^5F quadric

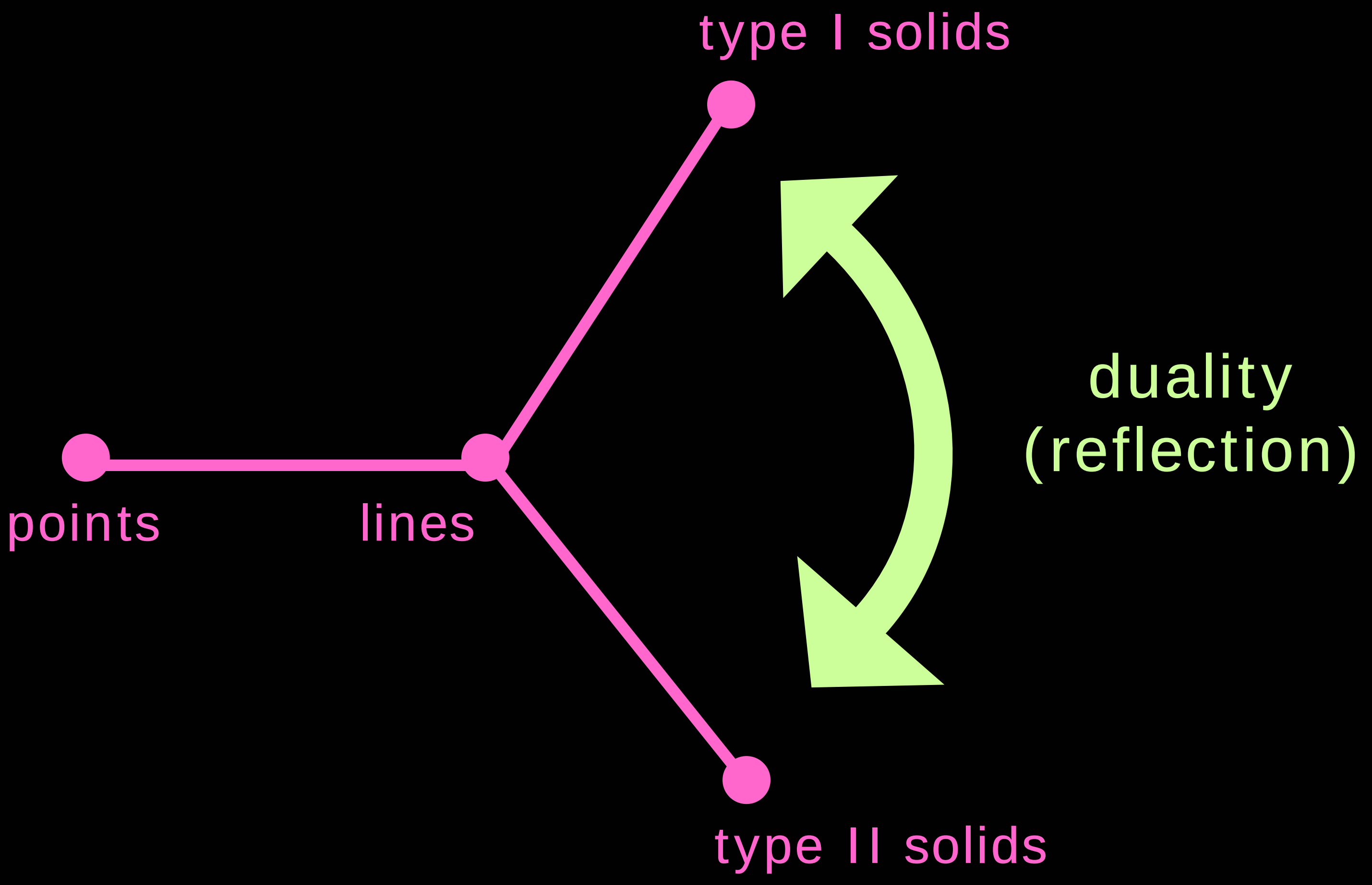


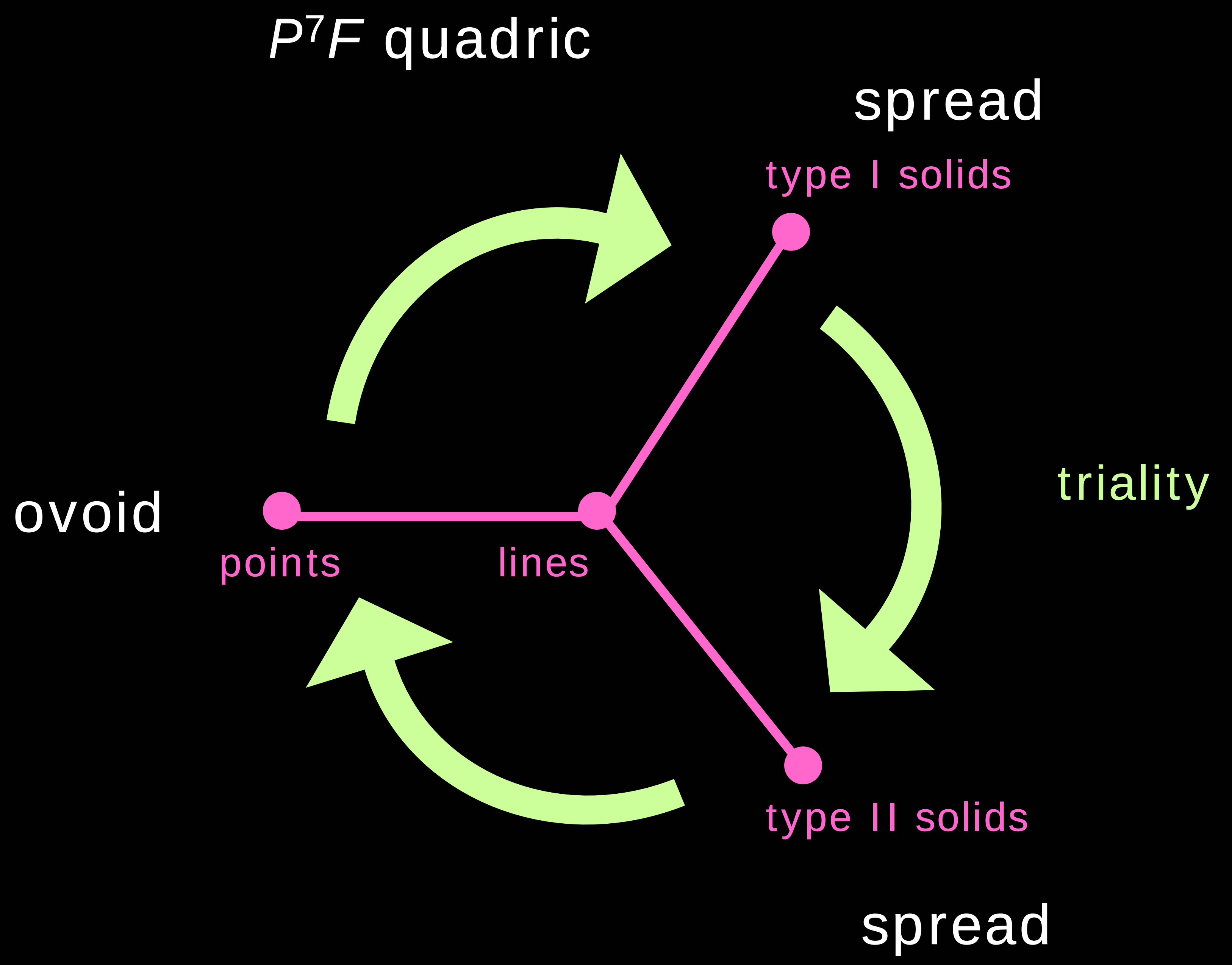




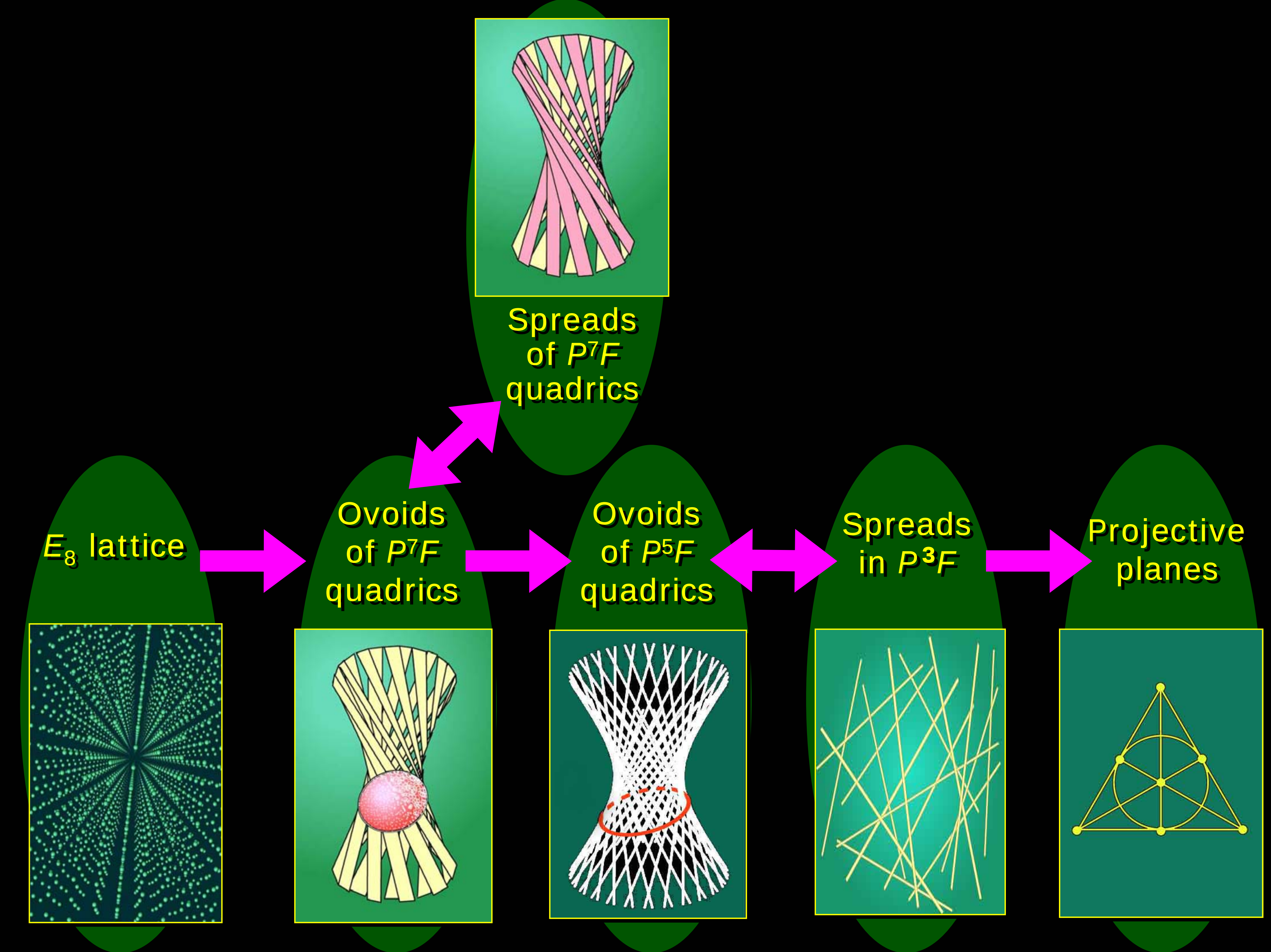


P^7F quadric


type I solids
duality
(reflection)
points
lines
type II solids



P^7F quadric
spread
type I solids
triality
ovoid
points
lines
type II solids
spread



Spreads of P^7F quadrics
E_8 lattice
Ovoids of P^7F quadrics
Ovoids of P^5F quadrics
Spreads in P^3F
Projective planes

Ovoids in $P^3\mathbb{F}_q$, $q=2^r$

Known examples:

- Elliptic quadrics admitting $PSL(2,q^2)$
- (r odd) Suzuki-Tits ovoids admitting ${}^2B_2(q)$

Code spanned by planes has dimension q^2+1.

Basis: $p^\perp$, $p \in \mathcal{O}$

$|\mathcal{O}| = q^2+1$

Ovoids in quadrics of $P^7\mathbb{F}_q$, $q=2^r$

Known examples:

- Examples admitting $PSL(3,q)$
- (r odd) Examples admitting $PSU(3,q)$
- ($q=8$) sporadic example

Code spanned by tangent hyperplanes to quadric has dimension q^3+1.

Basis: $p^\perp$, $p \in \mathcal{O}$

$|\mathcal{O}| = q^3+1$

Ovoids in quadrics of $P^6\mathbb{F}_q$, $q=3^r$

Known examples:

- Examples admitting $PSU(3,q)$
- (r odd) Ree-Tits ovoids admitting ${}^2G_2(q)$

Code spanned by tangent hyperplanes to quadric has dimension q^3+1.

Basis: $p^\perp$, $p \in \mathcal{O}$

$|\mathcal{O}| = q^3+1$

Ovoids in quadrics of $P^n\mathbb{F}_q$, $q=p^r$

- *always* exist for n=7 and r=1 (use E_8 root lattice) [J.H. Conway et. al. (1988); M. (1993)]
- do *not* exist for $p^{\lfloor n/2 \rfloor} > \binom{p+n-1}{n} - \binom{p+n-3}{n}$

  [Blokhuis and M. (1995)]

  e.g. ovoids do not exist

  - for n=9, p=2,3;
  - for n=11, p=2,3,5,7; etc.

Code spanned by tangent hyperplanes to quadric has dimension

$$\left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^r + 1$$

Subcode spanned by tangent hyperplanes to putative ovoid has dimension

$$|\mathcal{O}| = p^{\lfloor n/2 \rfloor r} + 1$$

Ovoids in quadrics of $P^n\mathbb{F}_q$, $q=p^r$

- *always* exist for n=7 and r=1 (use E_8 root lattice)
  [J.H. Conway et. al. (1988); M. (1993)]
- do *not* exist for $p^{\lfloor n/2 \rfloor} > \binom{p+n-1}{n} - \binom{p+n-3}{n}$

  [Blokhuis and M. (1995)]

  e.g. ovoids do not exist

  - for n=9, p=2,3;
  - for n=11, p=2,3,5,7; etc.
- do *not* exist for n=8,10,12,14,16,…
  [Gunawardena and M. (1997)]

Similar results for ovoids on Hermitian varieties
[M. (1996)]

2. p-ranks

$F = \mathbb{F}_q$, $q = p^r$

$N = (q^{n+1} - 1)/(q - 1)$ = number of points of $P^n F$

The code over $F = \mathbb{F}_q$ spanned by (characteristic vectors of) hyperplanes of $P^n F$ has dimension

$$\binom{p+n-1}{n}^r + 1$$

[Goethals and Delsarte (1968); MacWilliams and Mann (1968); Smith (1969)]

Stronger information: Smith Normal Form of point-hyperplane adjacency matrix [Black and List (1990)]

2. p-ranks

$F = \mathbb{F}_q$, $q = p^r$

$N = (q^{n+1} - 1)/(q - 1)$ = number of points of $P^n F$

More generally, let $\mathcal{C} = \mathcal{C}(n,k,p,r)$ be the code over F of length N spanned by projective subspaces of codimension k. Then

$$\dim \mathcal{C} = 1 + \left(\text{coeff. of } t^r \text{ in } \operatorname{tr}([I - tA]^{-1})\right)$$

where A is the $k \times k$ matrix with (i,j)-entry equal to the coefficient of t^{pj-i} in $(1 + t + t^2 + \ldots + t^{p-1})^{n+1}$.

Original formula for $\dim \mathcal{C}$ due to Hamada (1968).

This improved form is implicit in Bardoe and Sin (2000).

Smith Normal Form: Chandler, Sin and Xiang (2006).

2. p-ranks

$F=\mathbb{F}_q$, $q = p^r$

$\mathcal{Q}$: nondegenerate quadric in P^4F

$N = (q^4-1)/(q-1)$ = number of points of $\mathcal{Q}$

$\mathcal{C} = \mathcal{C}(n,p,r)$ = the code over $F=\mathbb{F}_q$ of length N spanned by (characteristic vectors of) lines which lie on $\mathcal{Q}$

$$\dim \mathcal{C} = \begin{cases} 1 + \left(\frac{1+\sqrt{17}}{2}\right)^{2r} + \left(\frac{1-\sqrt{17}}{2}\right)^{2r}, \quad p=2 & \text{[Sastry and Sin (1996)];} \\ 1 + \frac{p(p+1)^2}{2}, \quad q=p & \text{[de Caen and M. (1998)];} \\ 1 + \alpha^r + \beta^r; \quad \alpha,\beta = \frac{p(p+1)^2}{4} \pm \frac{p(p^2-1)}{12}\sqrt{17}, \quad q=p^r & \text{[Chandler, Sin and Xiang (2006)].} \end{cases}$$

3. Computing p-ranks via the Hilbert Function

Consider the $[N,k+1]$ code over $F=\mathbb{F}_q$ spanned by (characteristic vectors of) hyperplanes of P^nF.

$q = p^r$

N = number of points = $(q^{n+1}-1)/(q-1)$

$k = \binom{p+n-1}{n}^r$

The subcode $\mathcal{C}$ spanned by complements of hyperplanes has dimension k.

$\mathcal{V}$: subset of points of P^nF

$\mathcal{C}_{\mathcal{V}}$: the code of length $|\mathcal{V}|$ consisting of *puncturing*: restricting $\mathcal{C}$ to the points of $\mathcal{V}$

$\dim(\mathcal{C}_{\mathcal{V}})$ = ?

3. Computing *p*-ranks via the Hilbert Function

$F = \mathbb{F}_q$

$R = F[X_0, X_1, \ldots, X_n] = \bigoplus_{d \geq 0} R_d$, R_d = d-homogeneous part of R

Ideal $I \subseteq R$

F-rational points $\mathcal{V} = \mathcal{V}(I + J)$, $J = (X_i^q X_j - X_i X_j^q \ : 0 \leq i < j \leq n)$

$\mathcal{I} = \mathcal{I}(\mathcal{V}) \subseteq R$, $\mathcal{I}_d = \mathcal{I} \cap R_d$

Hilbert Function

$h_{\mathcal{I}}(d) = \dim(R_d / \mathcal{I}_d)$

= no. of standard monomials of degree d,
i.e. no. of monomials of degree d not in $\mathrm{LM}(\mathcal{I})$

Case $q=p$:

$\dim(\mathcal{C}_{\mathcal{V}}) = h_{\mathcal{I}}(p-1)$

3. Computing *p*-ranks via the Hilbert Function

$F = \mathbb{F}_q$

$R = F[X_0, X_1, \ldots, X_n] = \bigoplus_{d \geq 0} R_d$, $\quad R_d = d$-homogeneous part of R

Ideal $I \subseteq R$

F-rational points $\mathcal{V} = \mathcal{V}(I + J)$, $\; J = (X_i^q X_j - X_i X_j^q \; : 0 \leq i < j \leq n)$

$\mathcal{I} = \mathcal{I}(\mathcal{V}) \subseteq R$, $\; \mathcal{I}_d = \mathcal{I} \cap R_d$

Case $q = p^r$: Recall *Lucas' Theorem*. Write

$$c = c_0 + pc_1 + p^2 c_2 + \ldots ;$$
$$d = d_0 + pd_1 + p^2 d_2 + \ldots .$$

Then

$$\binom{d}{c} \equiv \prod_i \binom{d_i}{c_i} \mod p$$

Modified Hilbert Function:
$h^*_{\mathcal{I}}(d)$ = no. of monomials of the form $m_0 m_1^p m_2^{p^2} \ldots$ *such that* $d = d_0 + pd_1 + p^2 d_2 + \ldots$, $\deg(m_i) = d_i$ *and* m_i standard

$\dim(\mathcal{C}_{\mathcal{V}}) = h^*_{\mathcal{I}}(p-1)^r$

[M. (1997)]

3. Computing *p*-ranks via the Hilbert Function

Example: Nondegenerate Quadrics

$I = (Q)$, $Q(X_0, X_1, \ldots, X_n) \in R_2$ nondegenerate quadratic form

F-rational points of Quadric

$\mathcal{Q} = \mathcal{V}((Q) + J)$, $J = (X_i^q X_j - X_i X_j^q : 0 \leq i < j \leq n)$

$\mathcal{C}_{\mathcal{Q}}$ = code over F of length $|\mathcal{Q}|$ spanned by the hyperplane intersections with the quadric

$\dim(\mathcal{C}_{\mathcal{Q}}) = \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^r$ [Blokhuis and M. (1995)]

3. Computing *p*-ranks via the Hilbert Function

Example: Hermitian Variety

$F = \mathbb{F}_{q^2}$, $q = p^r$

$I = (U)$, $U(X_0, X_1, \ldots, X_n) = \sum_i X_i^{q+1} \in R_{q+1}$

F-rational points

$\mathcal{H} = \mathcal{V}((U) + J)$, $J = (X_i^{q^2} X_j - X_i X_j^{q^2} : 0 \le i < j \le n)$

$\mathcal{C}_{\mathcal{H}}$ = code over *F* of length $|\mathcal{H}|$ spanned by the hyperplane intersections with $\mathcal{H}$

$\dim(\mathcal{C}_{\mathcal{H}}) = \left[\binom{p+n-1}{n}^2 - \binom{p+n-2}{n}^2\right]^r$ [M. (1996)]

3. Computing p-ranks via the Hilbert Function

Example: Grassmann Varieties

$F = \mathbb{F}_q$, $q = p^r$

Plücker embedding:

$$\left\{\begin{array}{c}\text{projective}\\ s\text{-subspaces}\\ \text{of } P^mF\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{points of}\\ P^nF,\ n = \binom{m+1}{s+1} - 1\end{array}\right.$$

$I \subseteq R$ generated by homogeneous polynomials of degree 2 (van der Waerden syzygies)

F-rational points

$\mathcal{G} = \mathcal{V}(I + J)$, $J = (X_i^{\,q} X_j - X_i X_j^{\,q} : 0 \leq i < j \leq n)$

$\mathcal{C}_{\mathcal{G}}$ = code over F of length $|\mathcal{G}|$ spanned by the intersections of hyperplanes of P^nF with $\mathcal{G}$

$\dim(\mathcal{C}_{\mathcal{G}}) = h_{\mathcal{I}}(p-1)^r$, $h_{\mathcal{I}}(d) = \prod_{0 \leq j \leq s} \frac{(m+d-s+j)!\; j!}{(m-s+j)!\; (d+j)!}$ [M. (1997)]

3. Computing p-ranks via the Hilbert Function

Application: $F=\mathbb{F}_p$, $\mathcal{O}$ a conic in P^2F.

$\mathcal{C}$ = Code of length p^2+p+1 spanned by lines

Code spanned by lines		Code spanned by complements of lines
$\mathcal{C}$	$\supset$	$\mathcal{C}^\perp$
dimension $\binom{p+1}{2}+1$		dimension $\binom{p+1}{2}$

Obtain explicit basis for $\mathcal{C}^\perp$ using the $\binom{p+1}{2}$ secants to $\mathcal{O}$ and for C using the $\binom{p+1}{2}+1$ tangents and passants to $\mathcal{O}$.

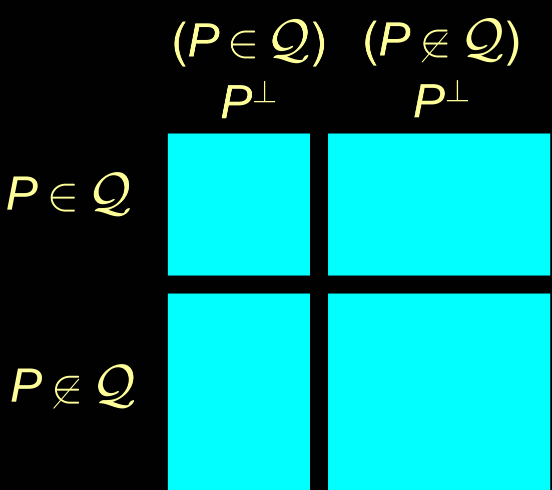
4. Open Problems

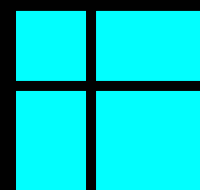
$F = \mathbb{F}_q$, $q = p^r$

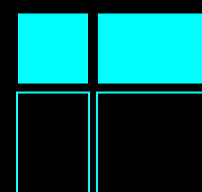
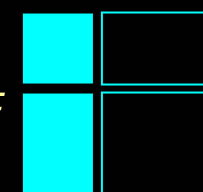
$N = (q^{n+1}-1)/(q-1)$ = number of points of $P^n F$

$\mathcal{Q}$: nondegenerate quadric in $P^n F$

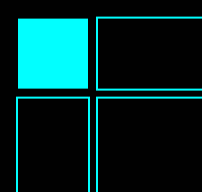
Point-hyperplane incidence matrix of $P^n F$:



rank_F  $= \binom{p+n-1}{n}^r + 1$

rank_F  $= \mathrm{rank}_F$ 

$$= \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^r + 1$$

rank_F  $= ?$

4. Open Problems

$F = \mathbb{F}_q$, $q = p^r$

$\mathcal{Q}$: nondegenerate quadric in P^nF

Can ovoids in $\mathcal{Q}$ exist for $n > 7$?

e.g. for $n = 23$ we require $p \geq 59$