Using the Borsuk-Ulam Theorem

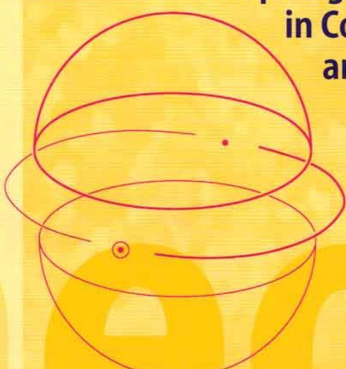
G Eric Moorhouse
University of Wyoming

based on Matoušek's book ...

Jiří Matoušek

Using the Borsuk-Ulam Theorem

Lectures on Topological Methods in Combinatorics and Geometry



Universitext

Springer

Jiří Matoušek, born in 1963, is Professor of Computer Science at Charles University in Prague. He works mainly in discrete geometry and combinatorics. This is his fourth book.

Using the Borsuk-Ulam Theorem

A number of important results in combinatorics, discrete geometry, and theoretical computer science have been proved by surprising applications of algebraic topology. While the results are quite famous, their proofs and the underlying methods are not so widely understood.

This textbook explains elementary but powerful topological methods based on the Borsuk-Ulam theorem and its generalizations.It covers many substantial results, sometimes with proofs simpler than those in the original papers. At the same time, it assumes no prior knowledge of algebraic topology, and all the required topological notions and results are gradually introduced. History, additional results, and references are presented in separate sections.

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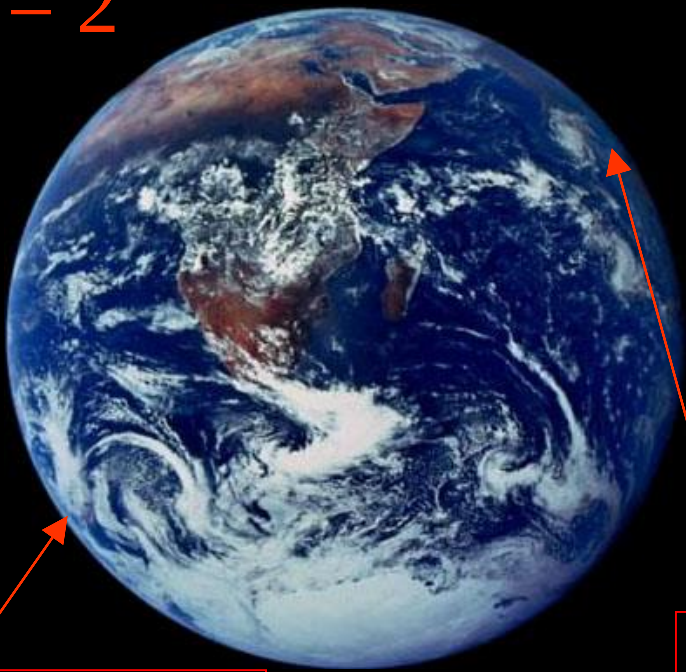
The Borsuk-Ulam Theorem

$n = 2$



$T = 69.154°C$
$P = 102.79\ kPa$

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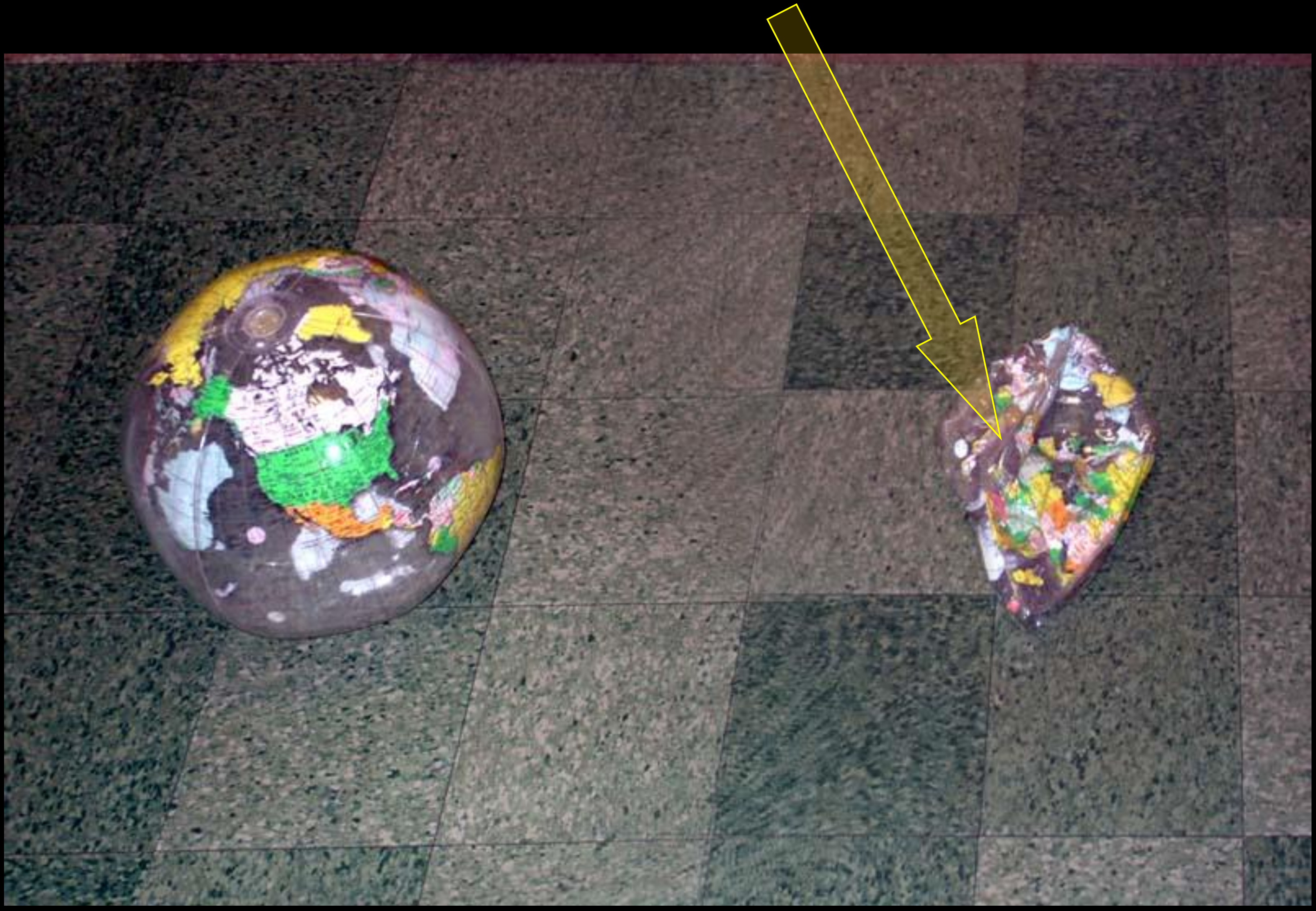
$P = 102.79\ kPa$

If $f : S^n \rightarrow \mathbb{R}^n$
is continuous then

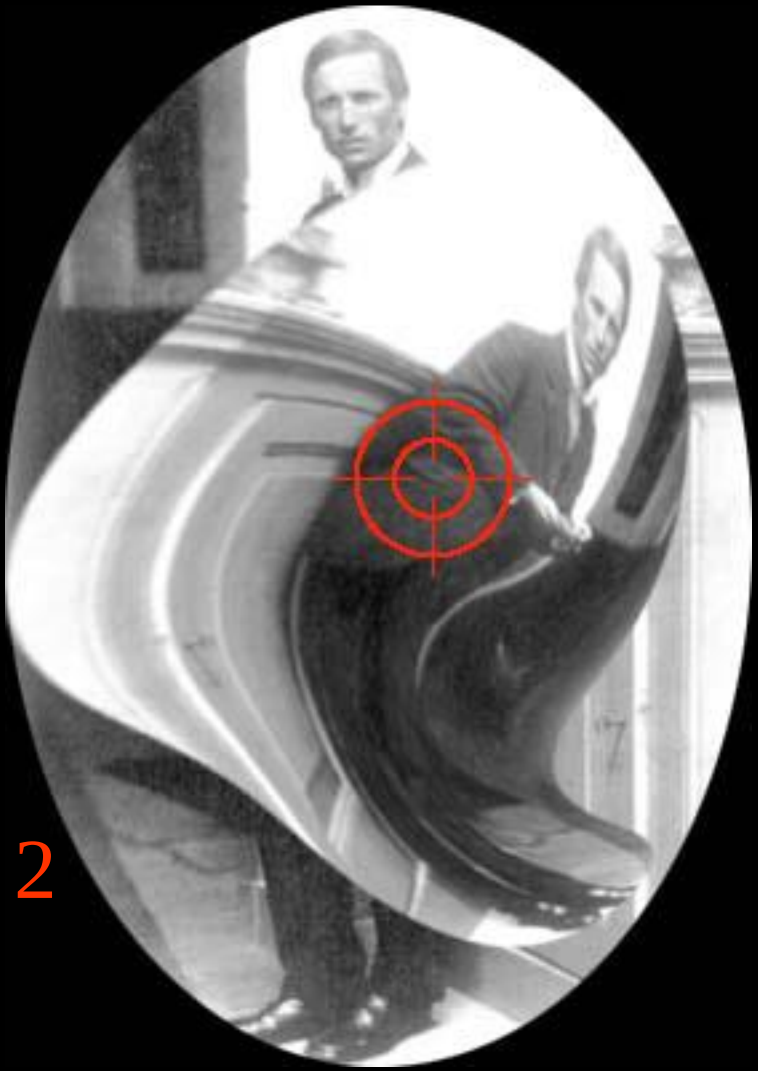
there exists $x \in S^n$
such that
$f(-x) = f(x)$.

In a deflated sphere, there is a point directly above its antipode.



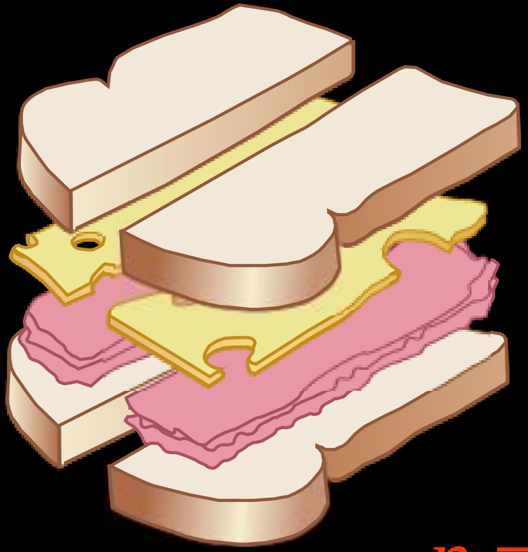
Brouwer Fixed-Point Theorem

If $f : B^n \to B^n$
then there exists
$x \in B^n$ such that
$f(x) = x$.



$n = 2$

The Ham Sandwich Theorem

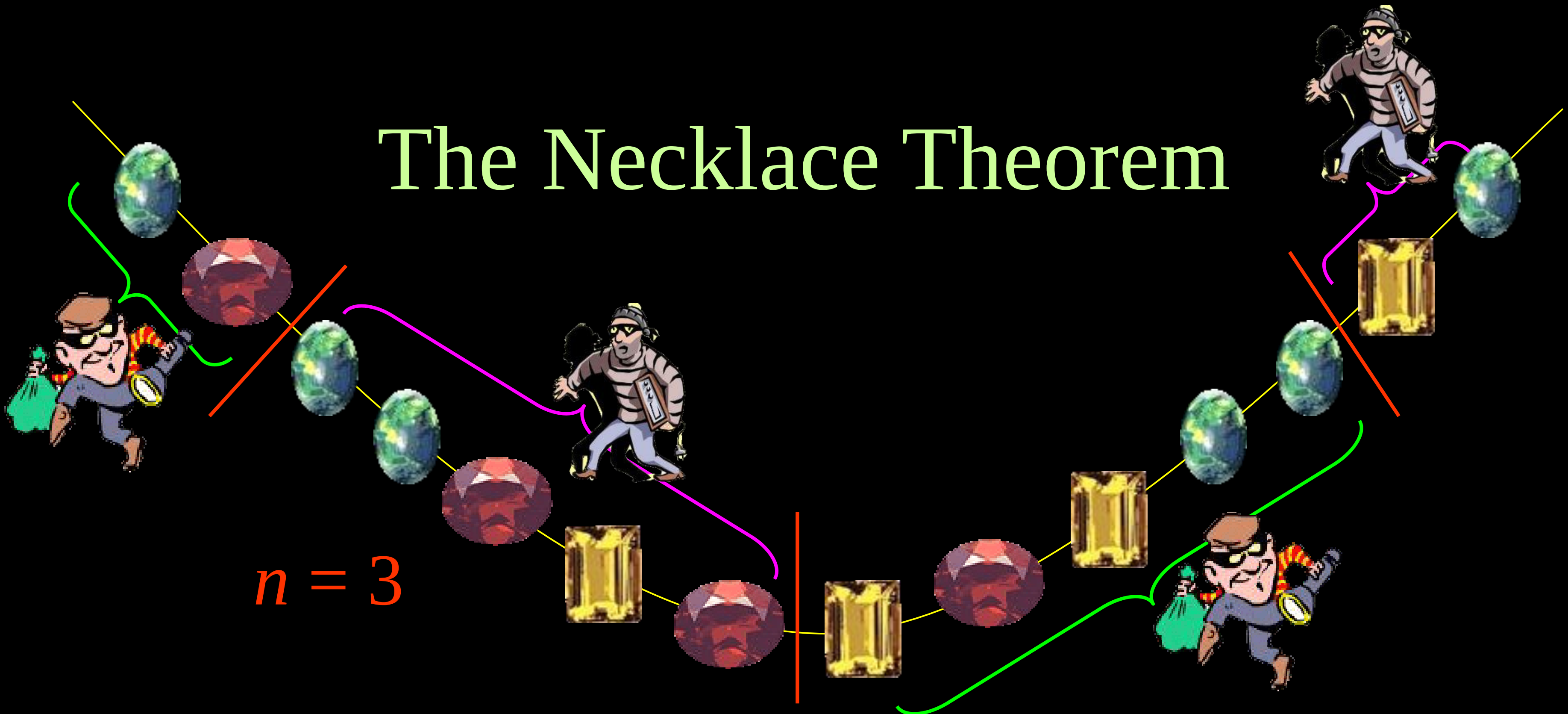


Given n mass distributions in $\mathbb{R}^n$, there exists a hyperplane dividing each of the masses.

$n = 3$

ham, cheese, bread

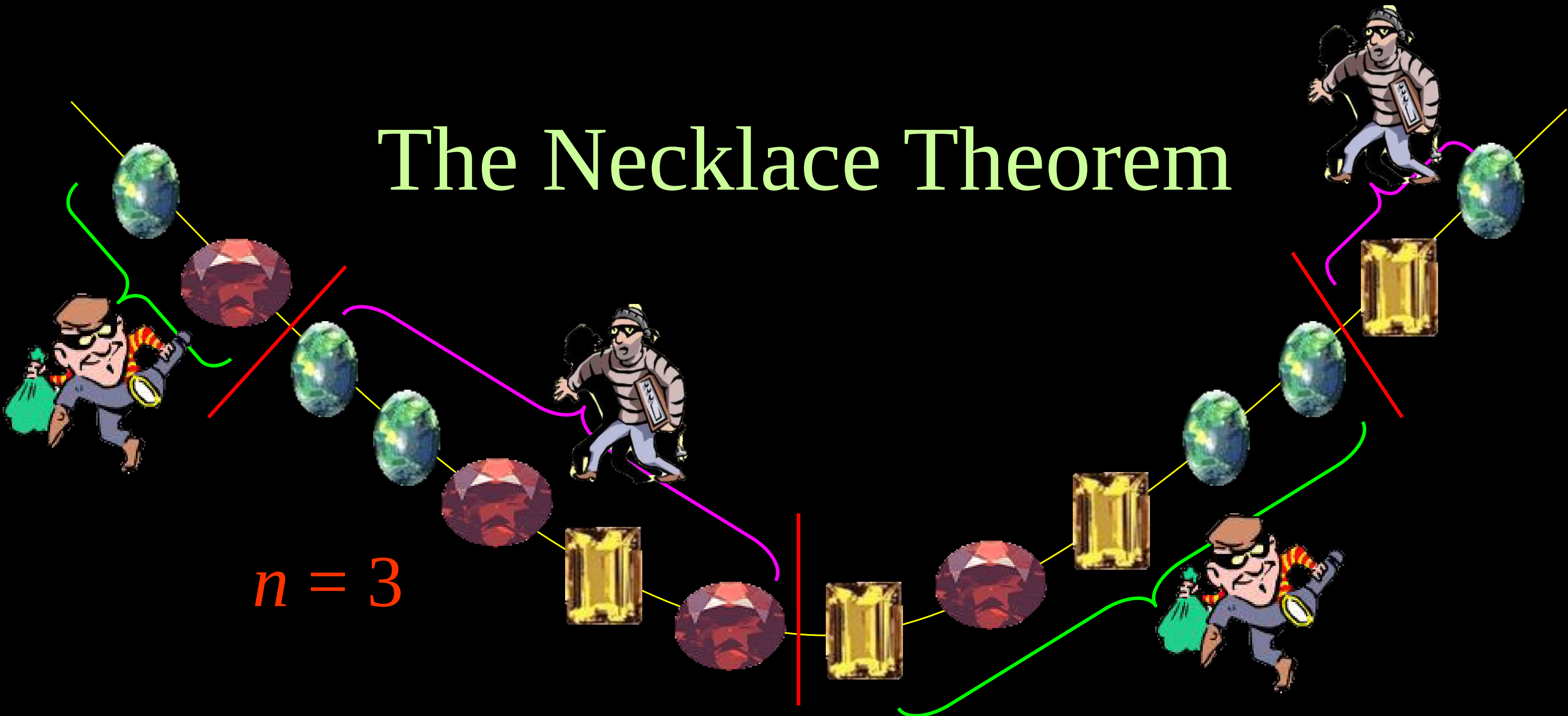
The Necklace Theorem



Every open necklace with *n* types of stones can be divided between two thieves using no more than *n* cuts.

There is a version for several thieves.

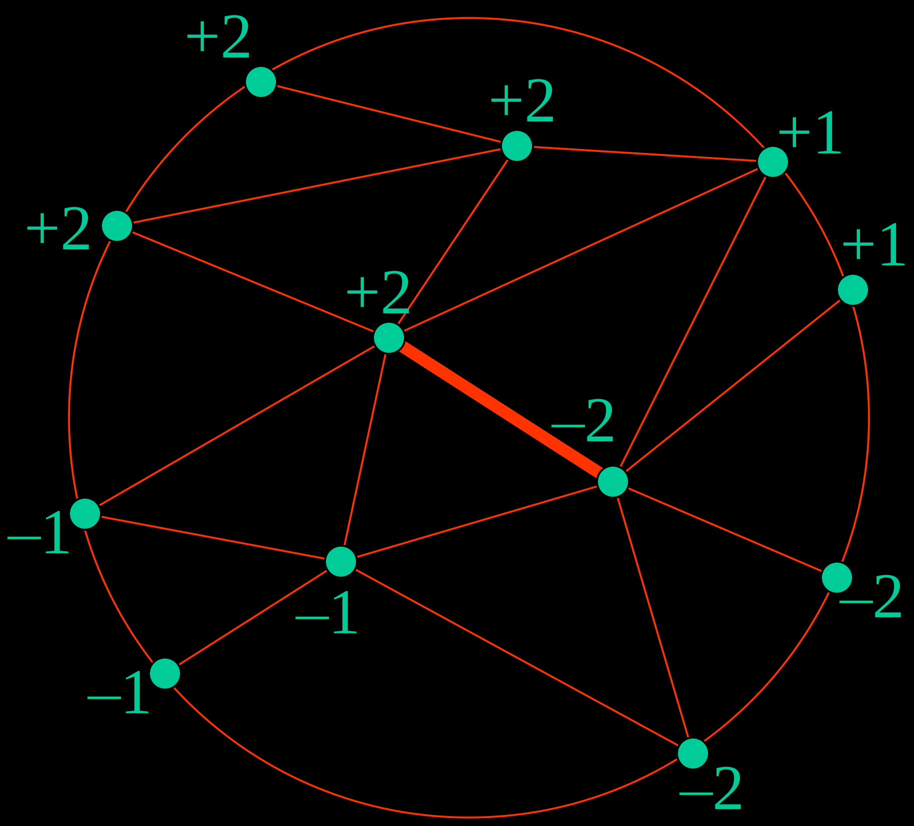
The Necklace Theorem



Every open necklace with n types of stones can be divided between two thieves using no more than n cuts.

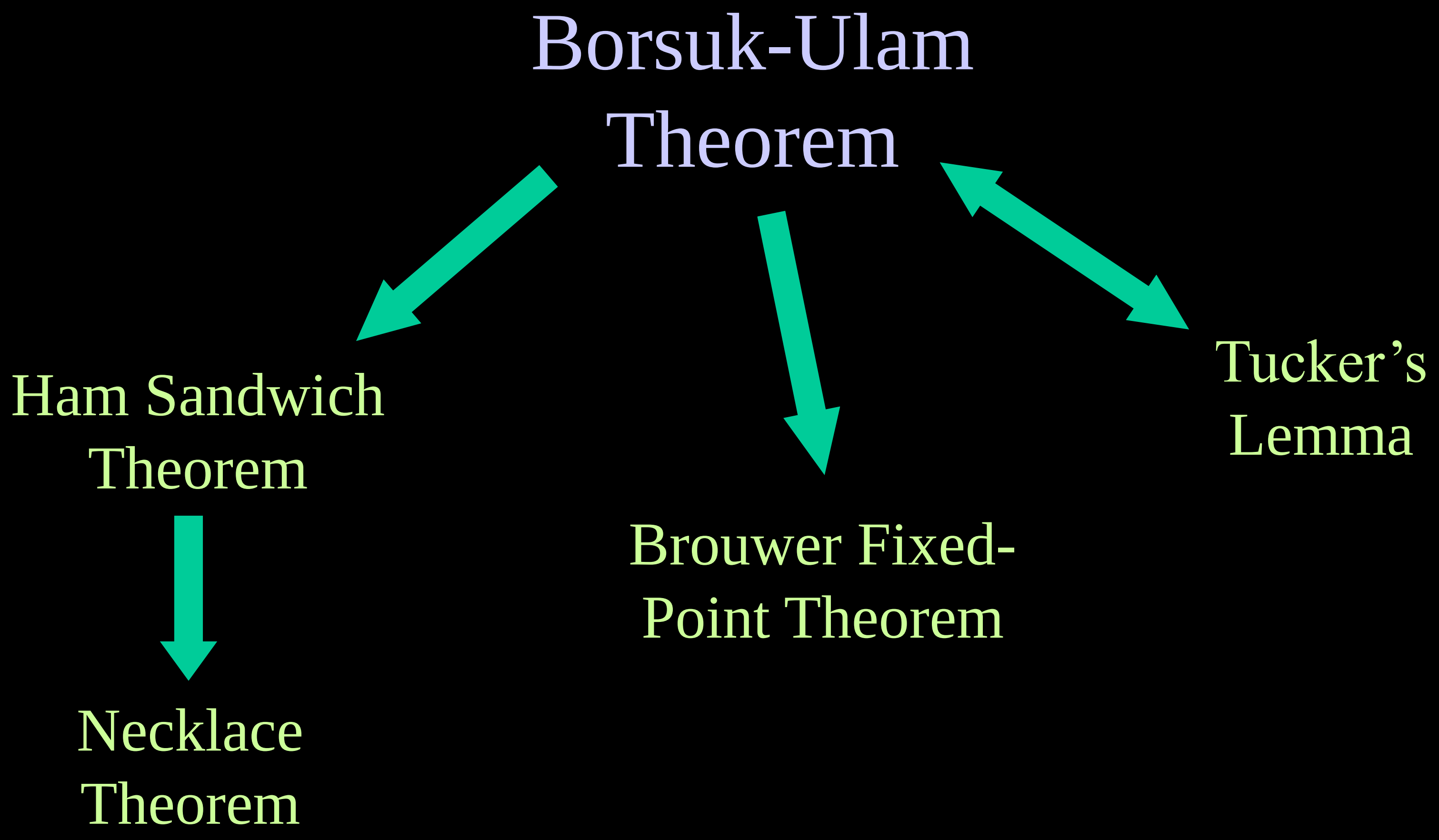
All known proofs are topological

Tucker's Lemma



$n = 2$

Consider a triangulation of B^n with vertices labeled $\pm 1, \pm 2, \ldots, \pm n$, such that the labeling is antipodal on the boundary. Then there exists an edge (1-simplex) whose endpoints have opposite labels $i, -i$.


Borsuk-Ulam Theorem
Ham Sandwich Theorem
Brouwer Fixed-Point Theorem
Tucker’s Lemma
Necklace Theorem

Versions of the Borsuk-Ulam Theorem

(1) (**Borsuk** 1933) If $f : S^n \to \mathbb{R}^n$ is continuous then there exists $x \in S^n$ such that $f(-x) = f(x)$.

(2) If $f : S^n \to \mathbb{R}^n$ is *antipodal*, i.e. $f(-x) = -f(x)$, then there exists $x \in S^n$ such that $f(x) = 0$.

(3) There is no antipodal map $S^n \to S^{n-1}$.

(4) (**Lyusternik-Schnirel'man** 1930) If $\{A_1, A_2, ..., A_{n+1}\}$ is a closed cover of S^n, then some A_i contains a pair of antipodal points.

(5) generalizing (4), each A_i is either open or closed

(Henceforth all maps are continuous functions.)

(1) If $f : S^n \to \mathbb{R}^n$ is continuous then there exists $x \in S^n$ such that $f(-x) = f(x)$.

$\Uparrow \Downarrow$

(2) If $f : S^n \to \mathbb{R}^n$ is *antipodal*, i.e. $f(-x) = -f(x)$, then there exists $x \in S^n$ such that $f(x) = 0$.

Let $f : S^n \to \mathbb{R}^n$ be antipodal.
There exists $x \in S^n$ such that $f(x) = f(-x) = -f(x)$. So $f(x) = 0$.

Let $f : S^n \to \mathbb{R}^n$ and define $g(x) = f(x) - f(-x)$.
Since g is antipodal, there exists $x \in S^n$ such that $g(x) = 0$.
So $f(-x) = f(x)$.

(1) (**Borsuk** 1933) If $f: S^n \to \mathbb{R}^n$ is continuous then there exists $x \in S^n$ such that $f(-x) = f(x)$.

$\Downarrow$

(4) (**Lyusternik-Schnirel'man** 1930) If $\{A_1, A_2, ..., A_{n+1}\}$ is a closed cover of S^n, then some A_i contains a pair of antipodal points.

Define $f: S^n \to \mathbb{R}^n$, $x \mapsto (\mathrm{dist}(x, A_1), ..., \mathrm{dist}(x, A_n))$.

There exists $x \in S^n$ such that $f(-x) = f(x) = y$, say.

If $y_i = 0$ $(i \le n)$ then $x, -x \in A_i$.

Otherwise $x, -x \in A_{n+1}$.

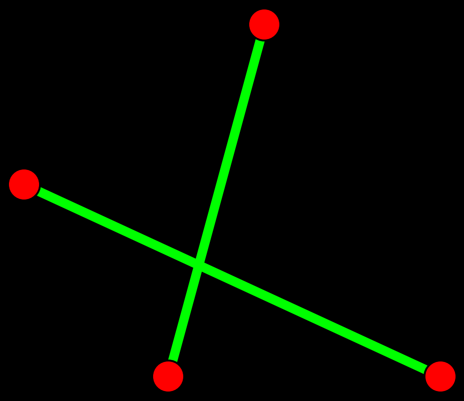
Radon's Theorem

Let $n \geq 1$.

Every set of $n+2$ points in $\mathbb{R}^n$ can be partitioned as $A_1 \cup A_2$ such that

$conv(A_1) \cap conv(A_2) \neq \emptyset$.

$n = 2$



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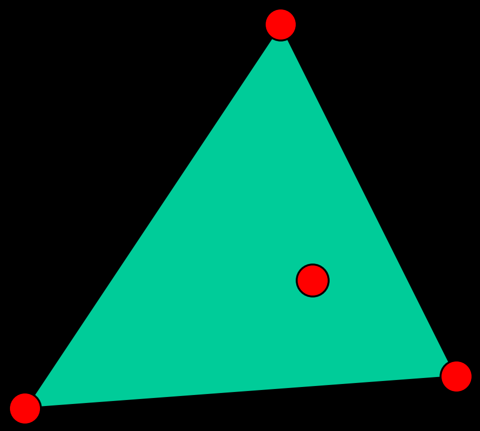
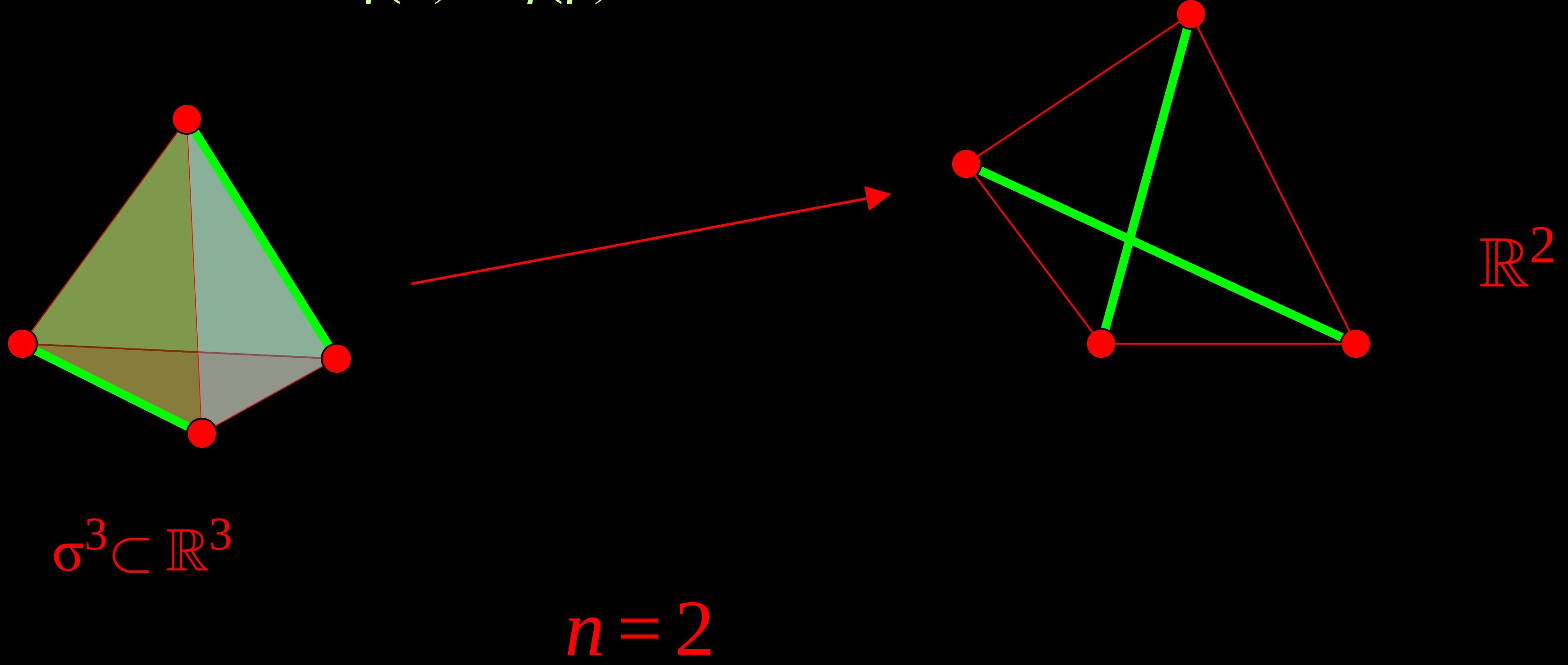


Radon's Theorem (Alternative Formulation)

Let σ^{n+1} be an n+1-simplex where $n \geq 1$ and let $f : \sigma^{n+1} \to \mathbb{R}^n$ be affine linear.

There exist two complementary sub-simplices α, β of σ^{n+1} such that $f(\alpha) \cap f(\beta) \neq \varnothing$.



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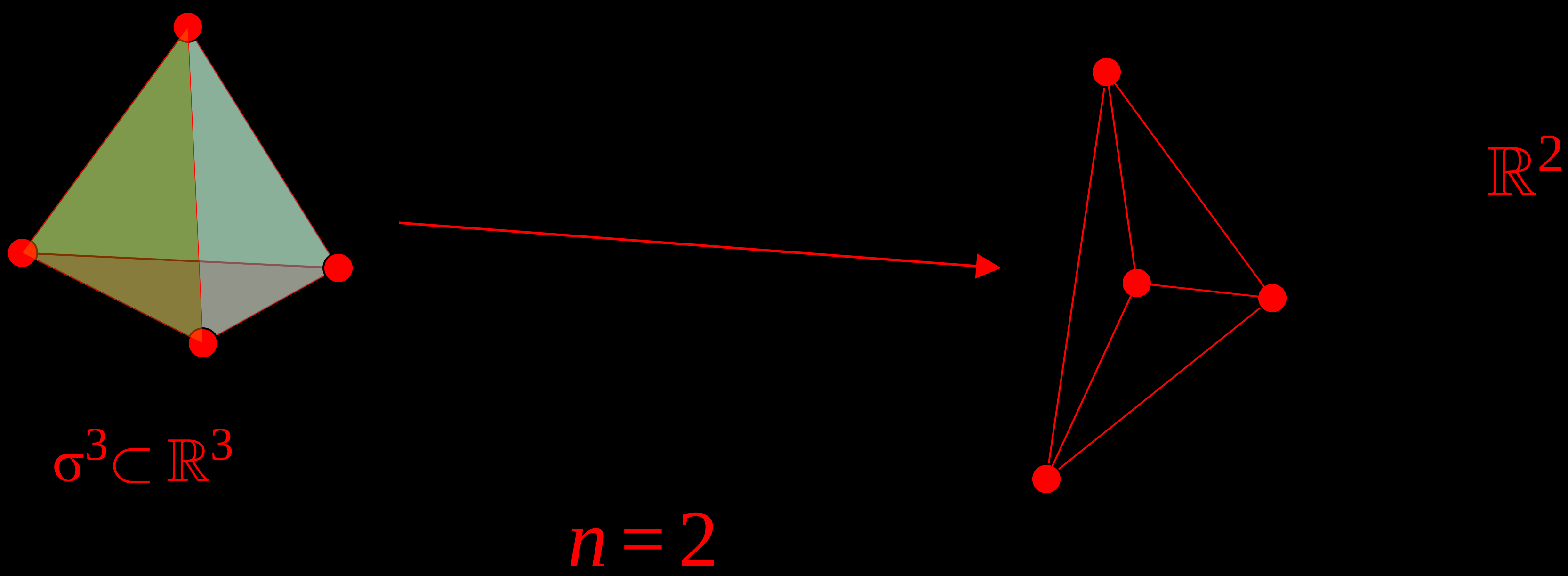

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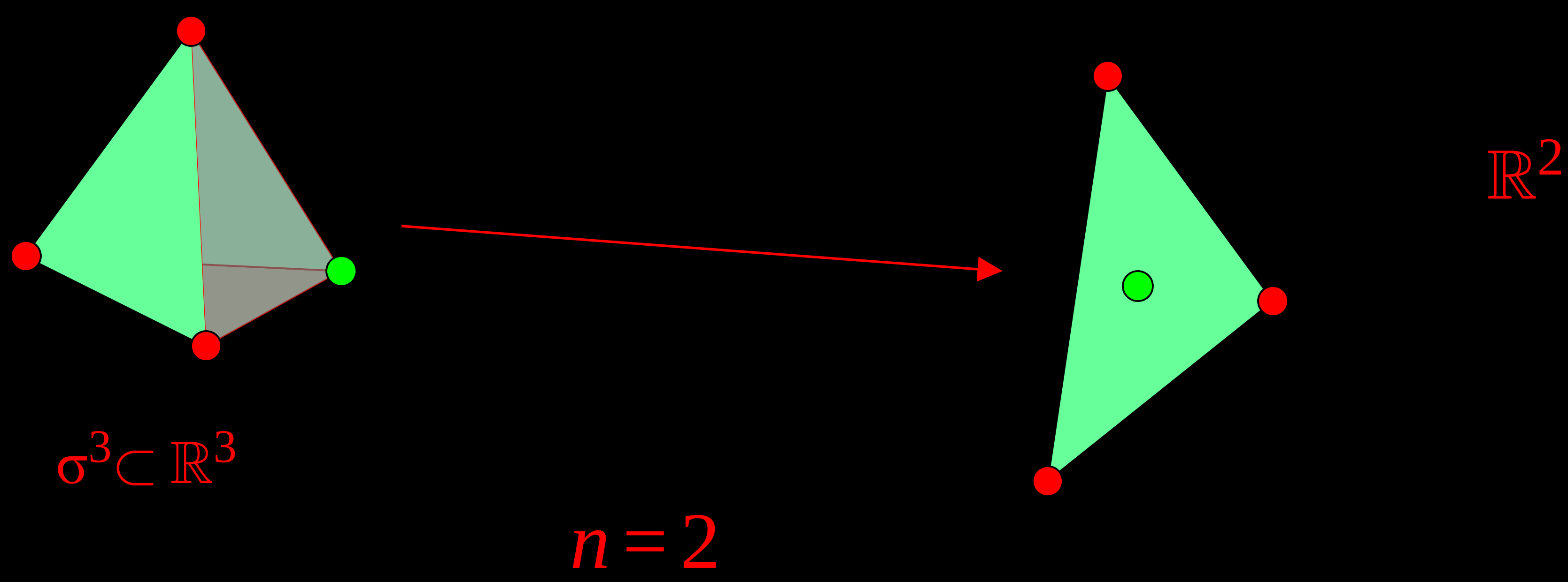

Topological Radon Theorem

Let σ^{n+1} be an n+1-simplex where $n \geq 1$ and let $f : \sigma^{n+1} \to \mathbb{R}^n$ be *continuous*.

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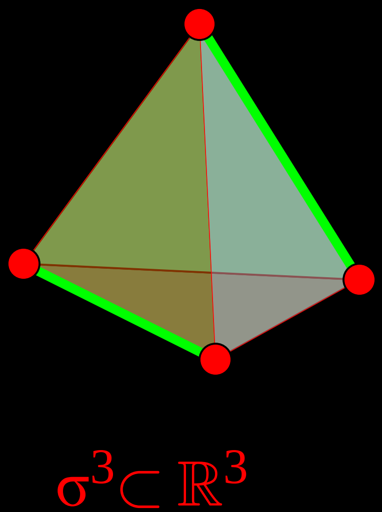

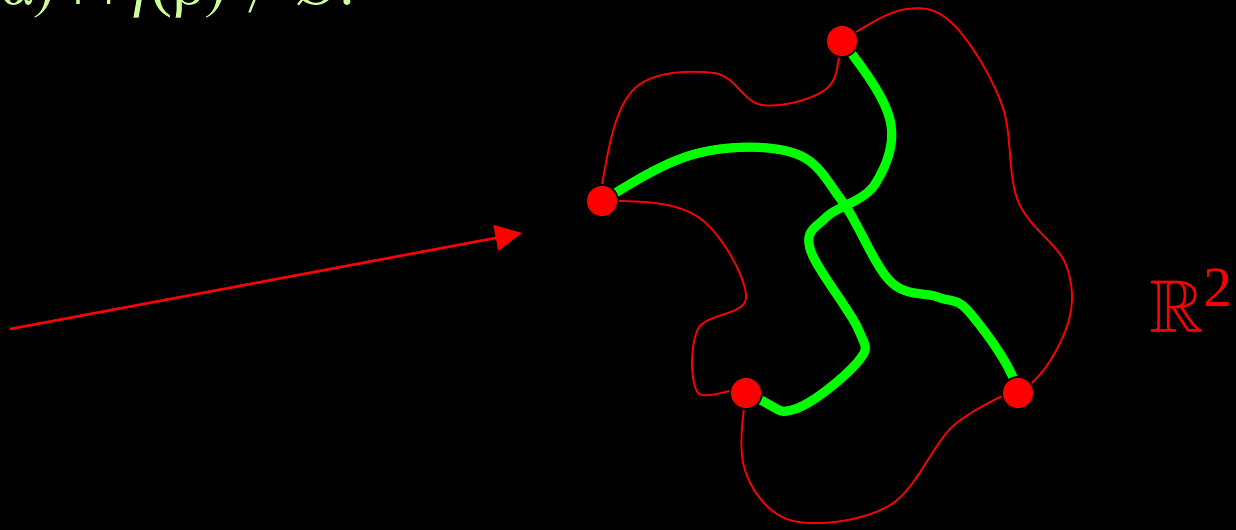

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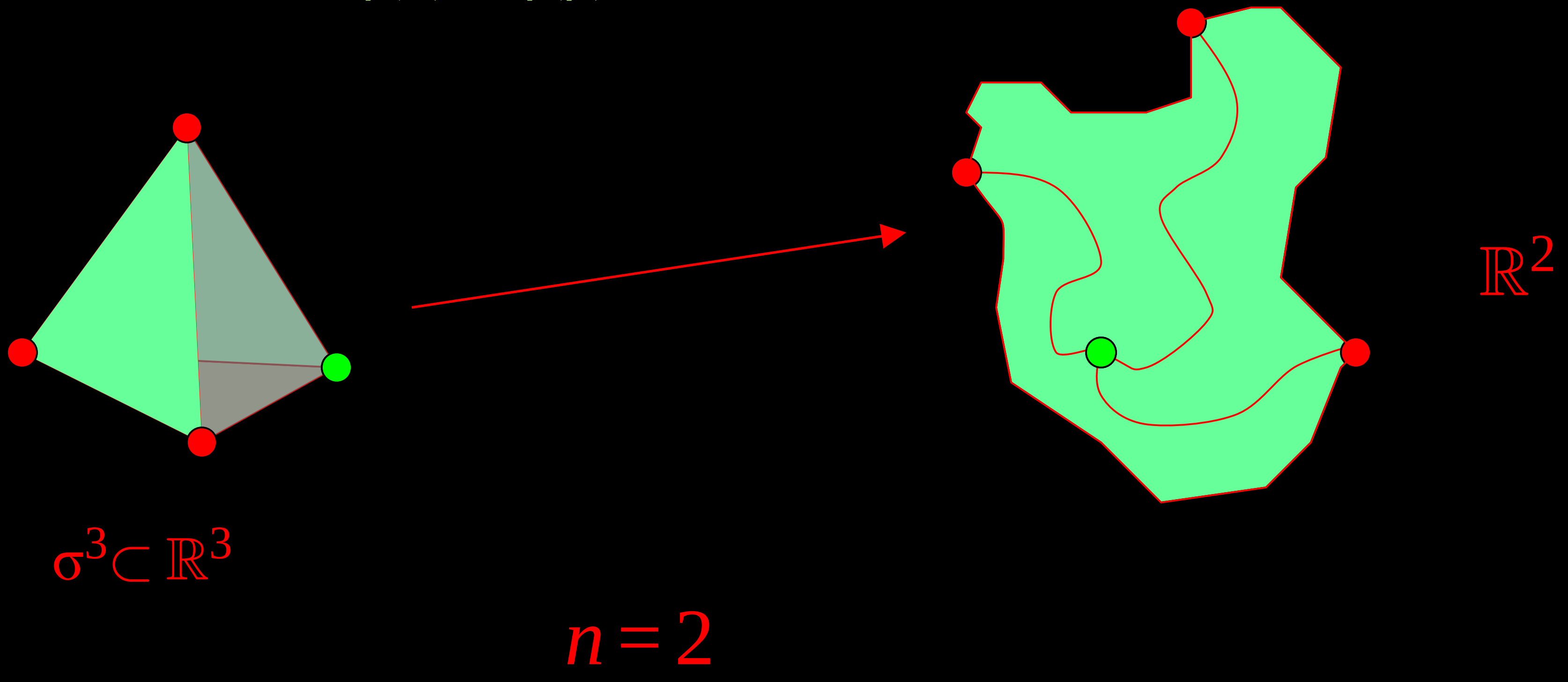


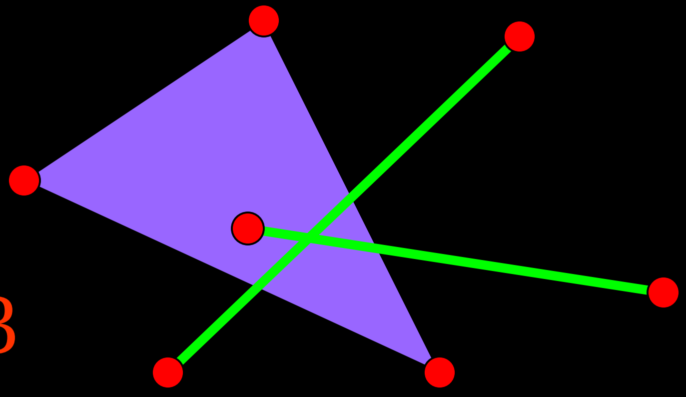
Tverberg's Theorem

Let $n \geq 1$, $r \geq 2$.

Every set of $nr+r-n$ points in $\mathbb{R}^n$
can be partitioned as $A_1 \cup A_2 \cup \dots \cup A_r$
such that

$conv(A_1) \cap conv(A_2) \cap \dots \cap conv(A_r) \neq \varnothing$.

$n = 2,\ r = 3$



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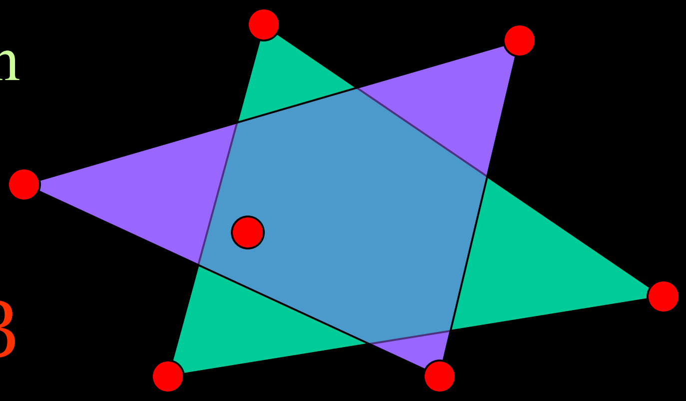
This generalization of Radon's Theorem
also has a valid topological version.

$n = 2,\ r = 3$



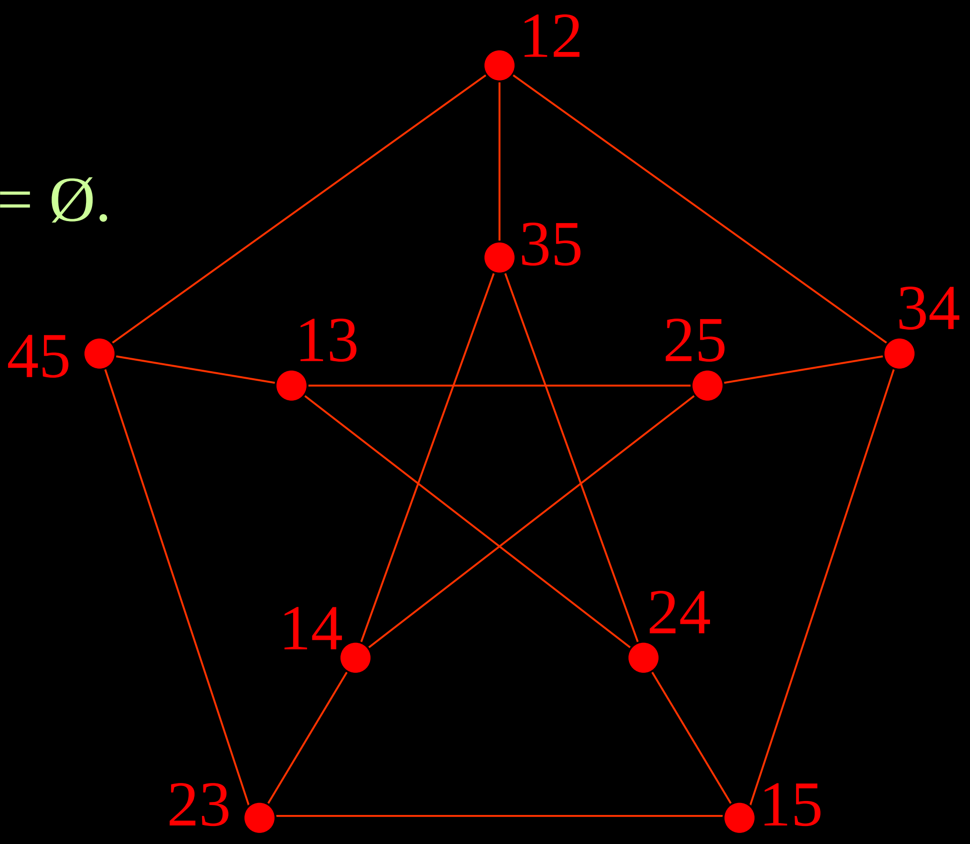
Lovász-Kneser Theorem

Kneser Graph $KG_{n,k}$ has $\binom{n}{k}$ vertices

$A \subseteq \{1, 2, \ldots, n\}, \ |A| = k.$

Here $1 \le k \le (n+1)/2$.

Vertices A, B are adjacent iff $A \cap B = \emptyset$.



$KG_{5,2}$

Lovász-Kneser Theorem

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$A \subseteq \{1, 2, \ldots, n\}, \ |A| = k.$

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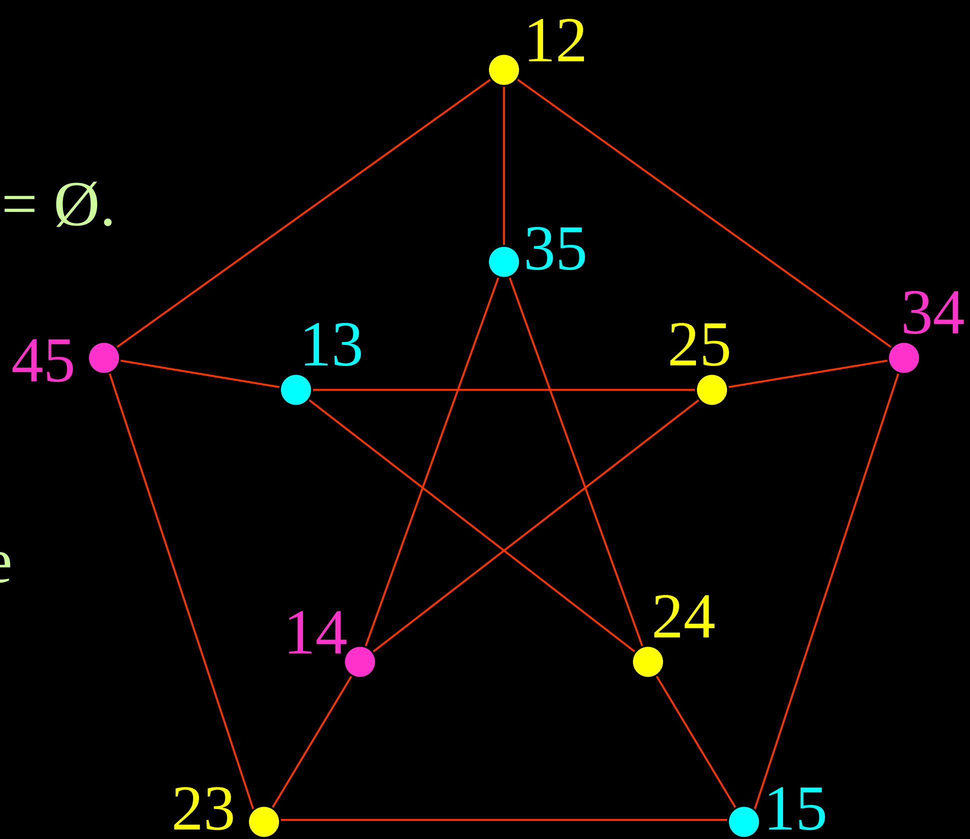
Vertices A,B are adjacent iff $A \cap B = \emptyset$.

Kneser Conjecture (1955)

$$\chi(KG_{n,k}) = n - 2k + 2.$$

Proved by Lovász (1978) using the Borsuk-Ulam Theorem.

The fractional chromatic number gives the very weak lower bound $\chi(KG_{n,k}) \geq n/k$.



$\chi(KG_{5,2}) = 3$

Lovász-Kneser Theorem

Kneser Graph $KG_{n,k}$ has $\binom{n}{k}$ vertices

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Here $1 \le k \le (n+1)/2$.

Vertices A,B are adjacent iff $A \cap B = \varnothing$.

A proper colouring of $KG_{n,k}$ with colours $1, 2, \ldots, n-2k+2$:

$$A \text{ is coloured: } \begin{cases} \min A \cap \{1,2,\ldots,n-2k+1\}, & \text{if this intersection is nonempty;} \\ n-2k+2 & \text{otherwise, i.e. } A \subseteq \{n-2k+2, \ldots, n\}. \end{cases}$$

Proof of Lovász-Kneser Theorem

Vertices of $KG_{n,k}$: k-subsets of an n-set $X \subset S^d$, $d = n-2k+1$.

WLOG points of X are in general position (no $d+1$ points on any hyperplane through 0).

Each $x \in S^d$ gives a partition $\mathbb{R}^{d+1} = H(x) \cup x^{\perp} \cup H(-x)$.

Suppose there is a proper colouring of $\binom{X}{k}$ using colours 1,2,...,d.

Define the point sets $A_1, A_2, \ldots, A_d \subseteq S^d$:

A_i is the set of all $x \in S^d$ for which some k-set $B \subseteq H(x)$ has colour i.

$A_{d+1} = S^d - (A_1 \cup A_2 \cup \ldots \cup A_d)$.

$A_1, A_2, \ldots, A_d$ are open; A_{d+1} is closed.

So some A_i contains a pair of antipodal points $x, -x$.

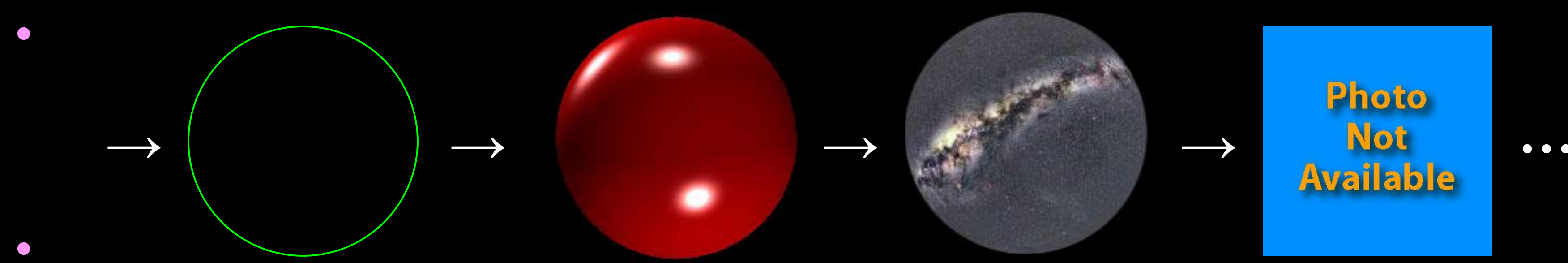
Case $i \le d$: we get k-tuples $A \subseteq H(x)$, $B \subseteq H(-x)$ of colour i. No!

Case $i = d+1$: $H(x)$ contains at most $k-1$ points of X. So does $H(-x)$.

So $x^{\perp}$ contains at least $n-2(k-1) = d+1$ points of X. No!

Similar techniques yield lower bounds for chromatic numbers for more general graphs using Z_2-indices ...

Sequence of spheres $S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$

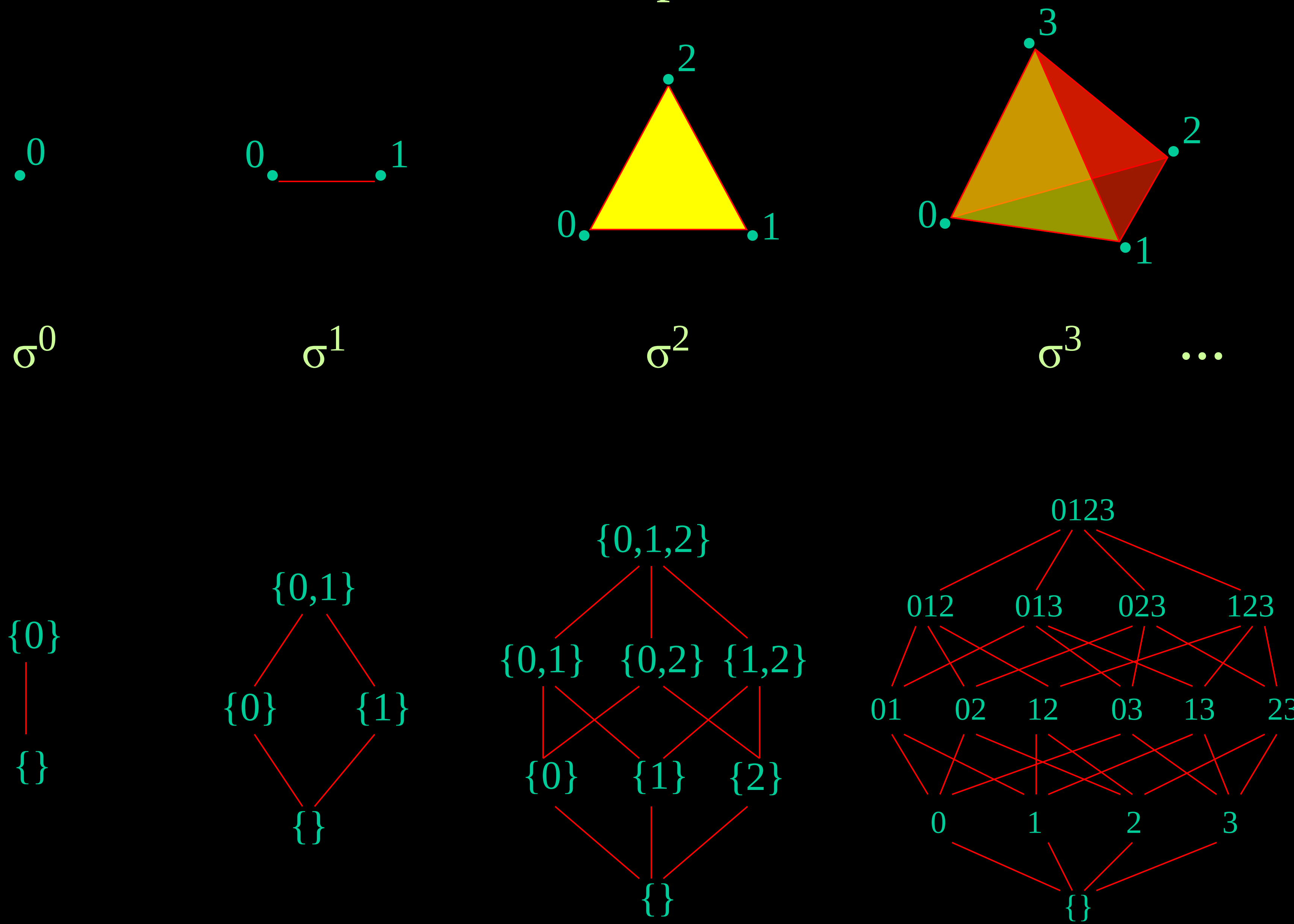


S^0 S^1 S^2 S^3 S^4

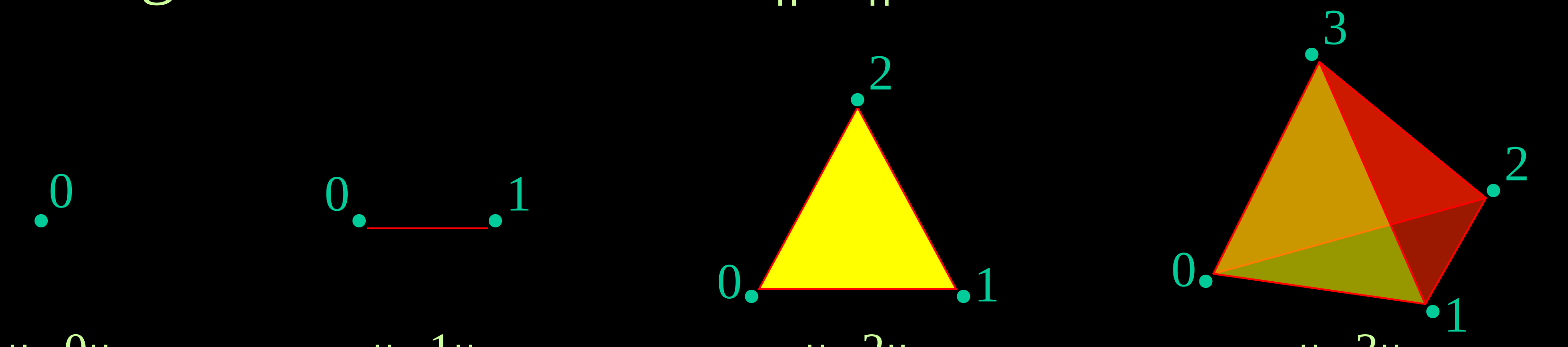
Antipodal maps $S^n \to S^{n+1}$ i.e. $f(-x) = -f(x)$

but $S^{n+1} \not\to S^n$

The n-simplex σ^n

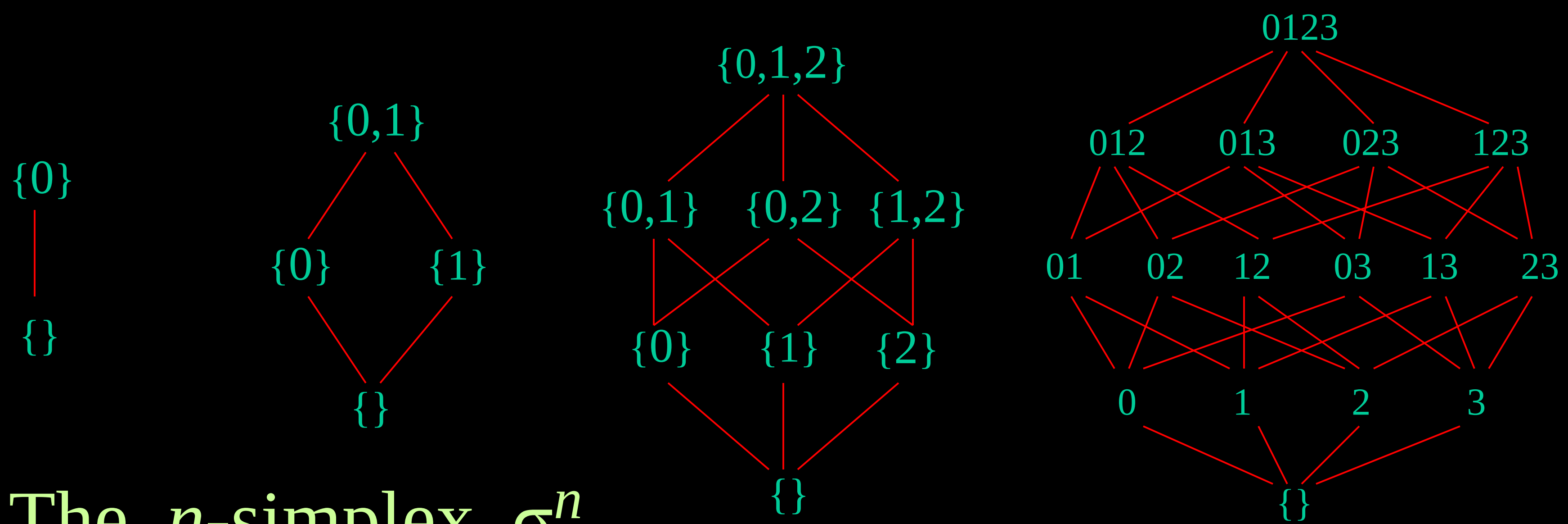


Its geometric realization $\|\sigma^n\| \subset \mathbb{R}^n$



$\|\sigma^0\|$ $\|\sigma^1\|$ $\|\sigma^2\|$ $\|\sigma^3\|$ ...

σ^0 σ^1 σ^2 σ^3

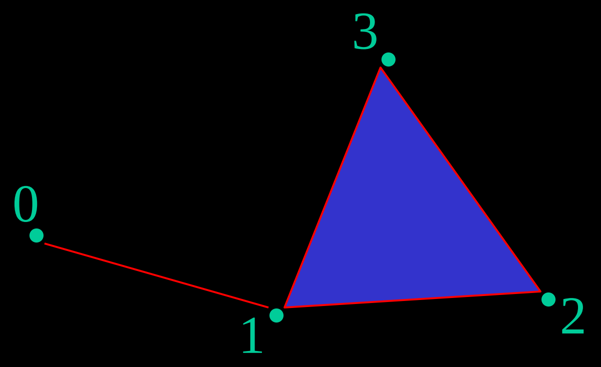


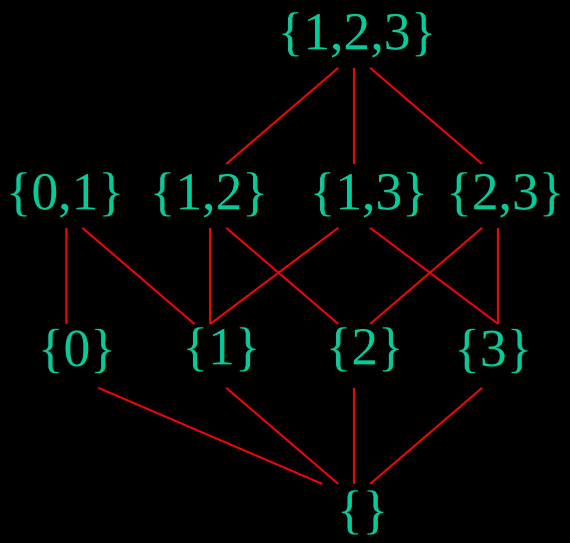
The n-simplex σ^n

A Simplicial Complex

e.g.

$||K||$ = geometric realization of K 　　　　 K

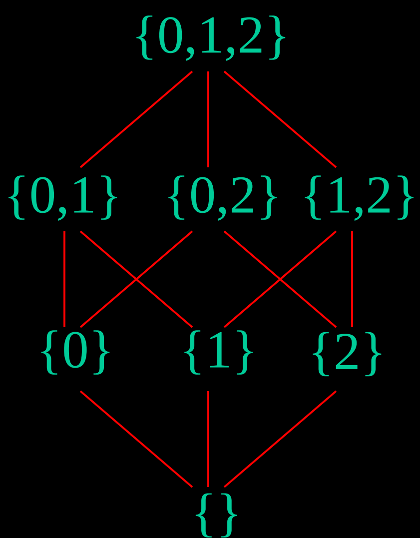




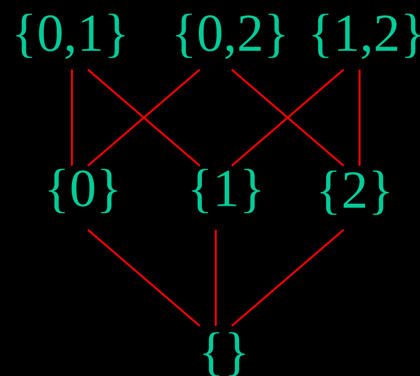
Skeletons

e.g. $K = \sigma^2$

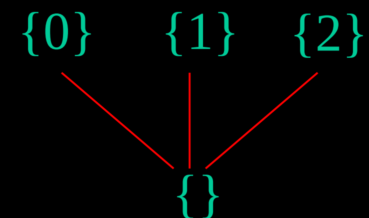
K



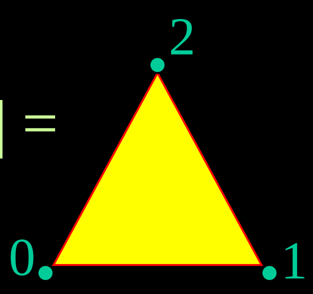
$K^{\leq 1}$
= the 1-skeleton of K



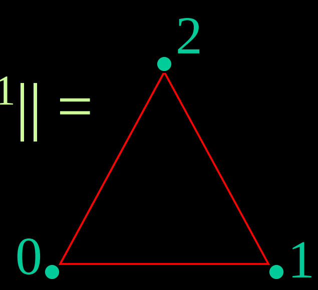
$K^{\leq 0}$
= the 0-skeleton of K



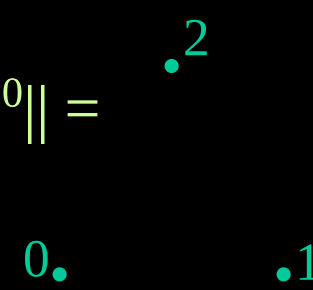
$\|K\| =$



$\|K^{\leq 1}\| =$



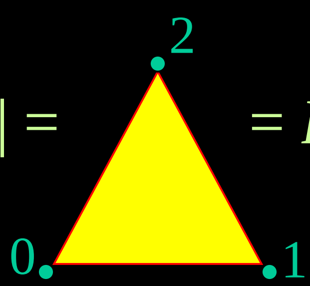
$\|K^{\leq 0}\| =$



$$\|\sigma^n\| = B^n, \qquad \|(\sigma^n)^{\leq n-1}\| = S^{n-1}, \qquad \|(\sigma^n)^{\leq 1}\| = K_{n+1}$$

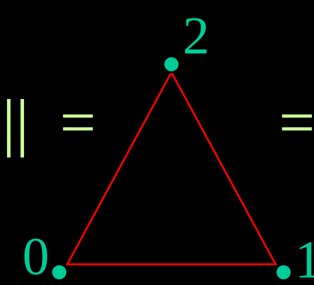
e.g.

$\|\sigma^2\| =$



$= B^2$

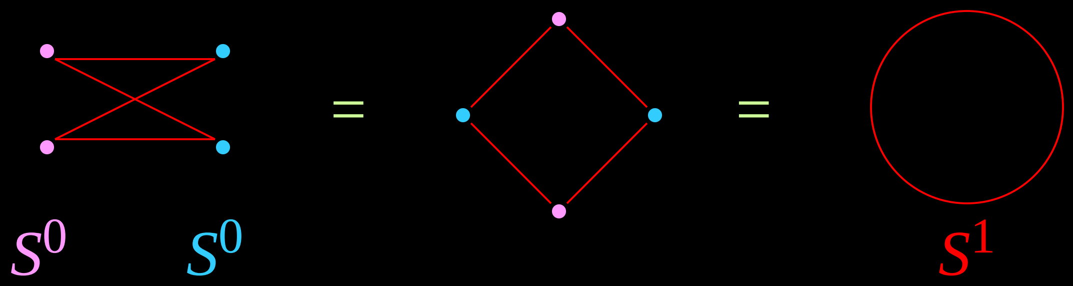
$\|(\sigma^2)^{\leq 1}\| =$



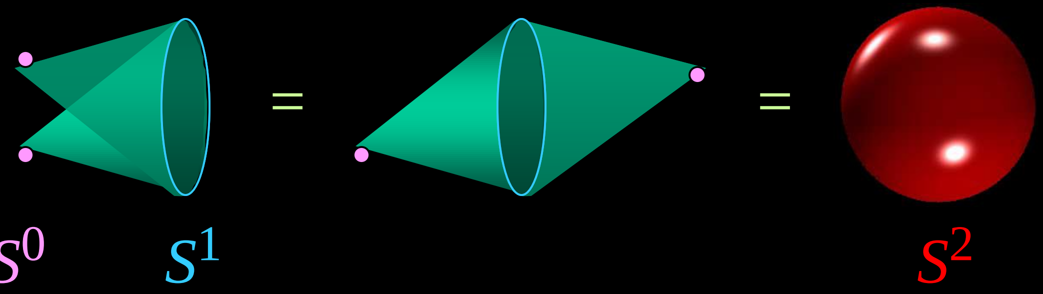
$= K_3 = S^2$

Topological join $S^n * S^m = S^{n+m+1}$

e.g. $S^0 * S^0 = S^1$



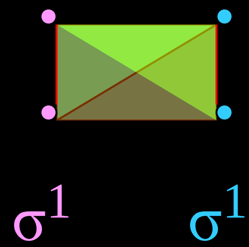
$S^0 * S^1 = S^2$



In particular $S^n = (S^0)^{*(n+1)} = S^0 * S^0 * \dots * S^0$

Join

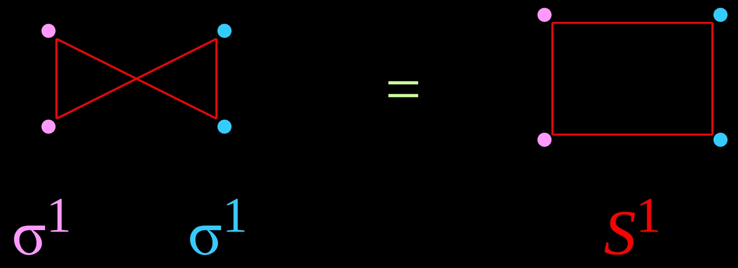
$$(\sigma^1)^{*2} = \sigma^1 * \sigma^1 = \sigma^3$$



More generally, $(\sigma^n)^{*2} = \sigma^{2n+1}$, $\|(\sigma^n)^{*2}\| = B^{2n+1}$.

Deleted Join

$$\|(\sigma^1)^{*2}_{\Delta}\| = S^1$$



More generally, $\|(\sigma^n)^{*2}_{\Delta}\| = S^n$.

Z_2-action on a topological space X:

a homeomorphism $X \longrightarrow X$, $x \mapsto x'$ such that $(x')' = x$
(not necessarily fixed-point-free). Denote $-x = x'$.

S^n and $\mathbb{R}^n$ have natural Z_2-actions.
The first is free, the second is not.

Let X and Y be topological Z_2-spaces. Write

$$X \longrightarrow Y$$

if there exists a Z_2-equivariant map $f : X \longrightarrow Y$, i.e. $f(-x) = -f(x)$.
If not, write $X \not\rightarrow Y$.

Thus $S^n \longrightarrow S^{n+1}$, $S^{n+1} \not\rightarrow S^n$.

If $X \longrightarrow Y$ and $Y \longrightarrow W$, then $X \longrightarrow W$.
So '$\longrightarrow$' defines a partial order.

Z_2-index and coindex of X:

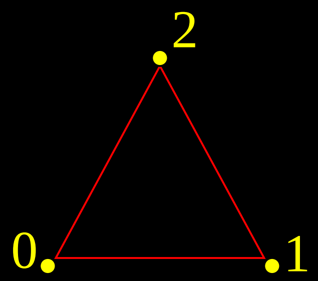
$\text{ind}_2(X) =$ smallest n such that $X \to S^n$;
$\text{coind}_2(X) =$ largest n such that $S^n \to X$.

Properties:

- If $\text{ind}_2(X) > \text{ind}_2(Y)$ then $X \nrightarrow Y$.
- $\text{coind}_2(X) \leq \text{ind}_2(X)$
- $\text{ind}_2(S^n) = \text{coind}_2(S^n) = n$
- $\text{ind}_2(X*Y) \leq \text{ind}_2(X) + \text{ind}_2(Y) + 1$
- If X is n–1-connected then $\text{ind}_2(X) \geq n$.
- If X is a *free* simplicial Z_2-complex (or cell Z_2-complex) of dimension n, then $\text{ind}_2(X) \leq n$.

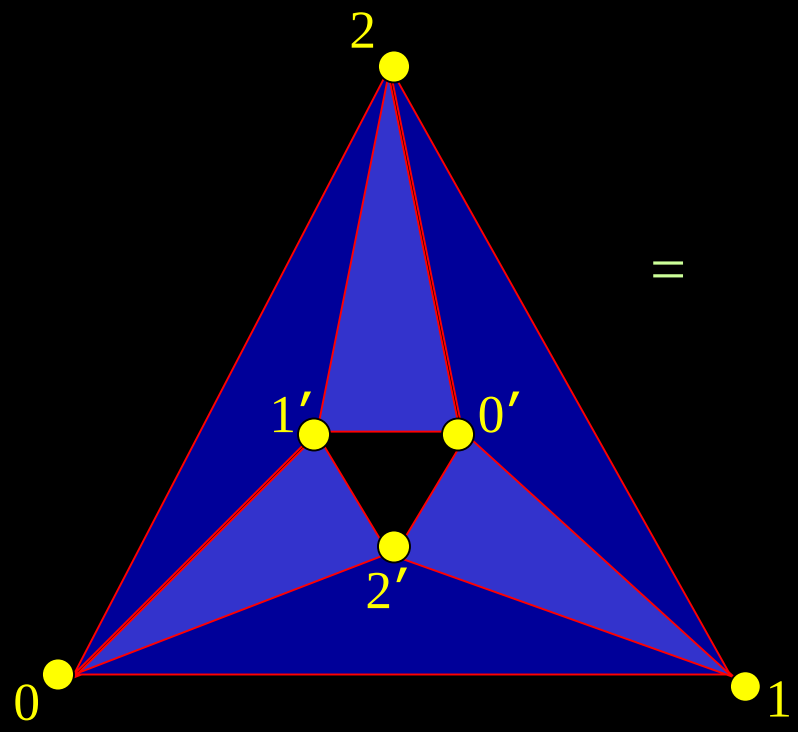
The Box Complex $B(\Gamma)$ of a Graph Γ

e.g. $\Gamma = K_3$



$\chi(\Gamma) = 3$

$\|B(\Gamma)\| = [0,1] \times S^1$



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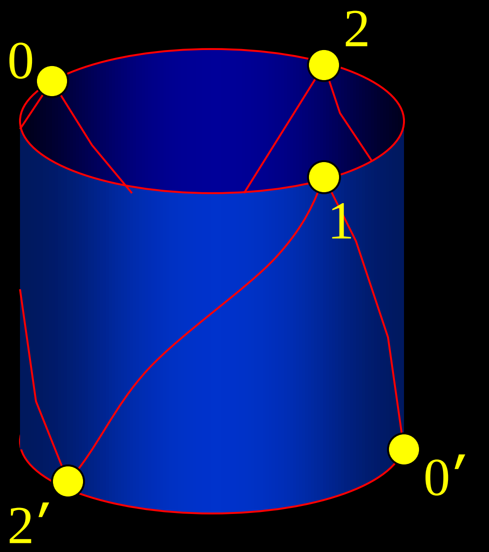
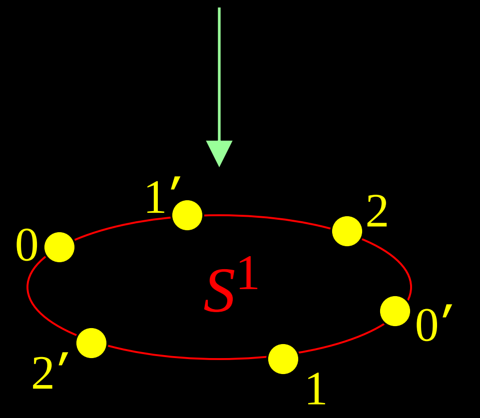




$\text{ind}_2(\|B(\Gamma)\|) = 1$

$\chi(\Gamma) \geq \text{ind}_2(\|B(\Gamma)\|) + 2$

The Box Complex $B(\Gamma)$ of a Graph Γ

$B(\Gamma)$ is the set of all pairs (A, B), $A, B \subseteq V(\Gamma)$ such that every member of A is adjacent to every member of B.

We allow $A = \emptyset$, but in this case we require that B has a nonempty set of common neighbours.

Similarly if $B = \emptyset$, we require that A has a nonempty set of common neighbours.

Nonembeddability of Deleted Join

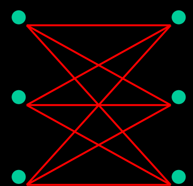
Let K be a simplicial complex. If

$$\mathrm{ind}_2(\,\|K\|_\Delta^{*2}\,) > n$$

then for every $f : \|K\| \to \mathbb{R}^n$, there exist two disjoint faces of K whose images in $\mathbb{R}^n$ intersect.

In particular, $\|K\|$ is not embeddable in $\mathbb{R}^n$.

Special case: the Topological Radon Theorem.

Another special case: $K = K_{3,3} =$  $=$ {  } $*$ {  }

$\mathrm{ind}_2(\,\|K\|_\Delta^{*2}\,) = 3$ so $K_{3,3}$ is nonplanar (i.e. nonembeddable in $\mathbb{R}^2$).

Another special case: $P^2\mathbb{R}$ is not embeddable in $\mathbb{R}^3$.

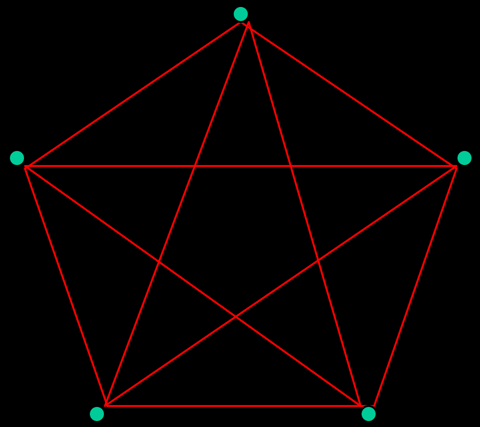
Van Kampen-Flores Theorem

Let $K = (\sigma^{2n+2})^{\leq n}$ where $n \geq 1$ (the n-skeleton of a $2n+2$-simplex).
Then $||K||$ is not embeddable in $\mathbb{R}^{2n}$.

Moreover:

For every map $f : ||\mathrm{K}|| \to \mathbb{R}^{2n}$, there exist two disjoint faces α,β of $||K||$ such that $f(\alpha) \cap f(\beta) \neq \emptyset$.

Case $n = 1$: $K = (\sigma^4)^{\leq 1} = K_5$ is not embeddable in $\mathbb{R}^2$.



Replace Z_2 by a (finite) group G

G acts freely on G (a discrete topological space with $|G|$ points).

Replace $S^n = (S^0)^{*(n+1)}$ by $G^{*(n+1)}$.

Consider topological spaces with G-action (not necessarily free).

Write $X \longrightarrow Y$ if there exists a G-equivariant map $f : X \longrightarrow Y$.

$\mathrm{ind}_G(X)$ = largest n such that $X \longrightarrow G^{*(n+1)}$;
$\mathrm{coind}_G(X)$ = smallest n such that $G^{*(n+1)} \longrightarrow X$.

Usually take $G = Z_p$ (cyclic of order p).

This gives a proof of the Topological Tverberg Theorem (generalizing the proof of the Topological Radon Theorem).

Proof of the Borsuk-Ulam Theorem

Suppose $f : S^n \to S^{n-1}$, $f(-x) = -f(x)$. Then f induces maps

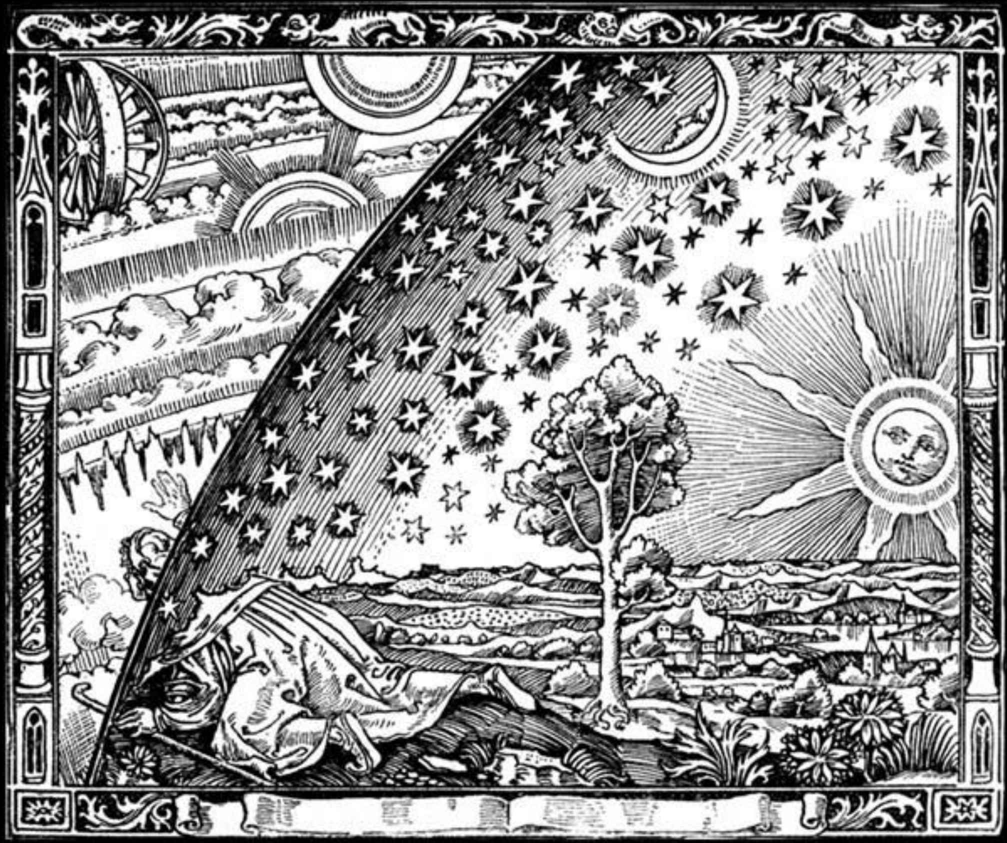
$$P^n\mathbb{R} \to P^{n-1}\mathbb{R}$$

$$\begin{array}{ccc} \pi_1(P^n\mathbb{R}) & \xrightarrow{\cong} & \pi_1(P^{n-1}\mathbb{R}) \\ \wr\| & & \wr\| \\ Z_2 & \xrightarrow{\cong} & Z_2 \end{array}$$

$$\begin{array}{ccc} H^*(P^{n-1}\mathbb{R}, \mathbb{F}_2) & \to & H^*(P^n\mathbb{R}, \mathbb{F}_2) \\ \wr\| & & \wr\| \\ \mathbb{F}_2[X]/(X^n) & \to & \mathbb{F}_2[X]/(X^{n+1}) \\ X & \mapsto & X, \end{array}$$

a contradiction.

The End



"Flat Earth" woodcut, 1888 (Flammarion)