Knots

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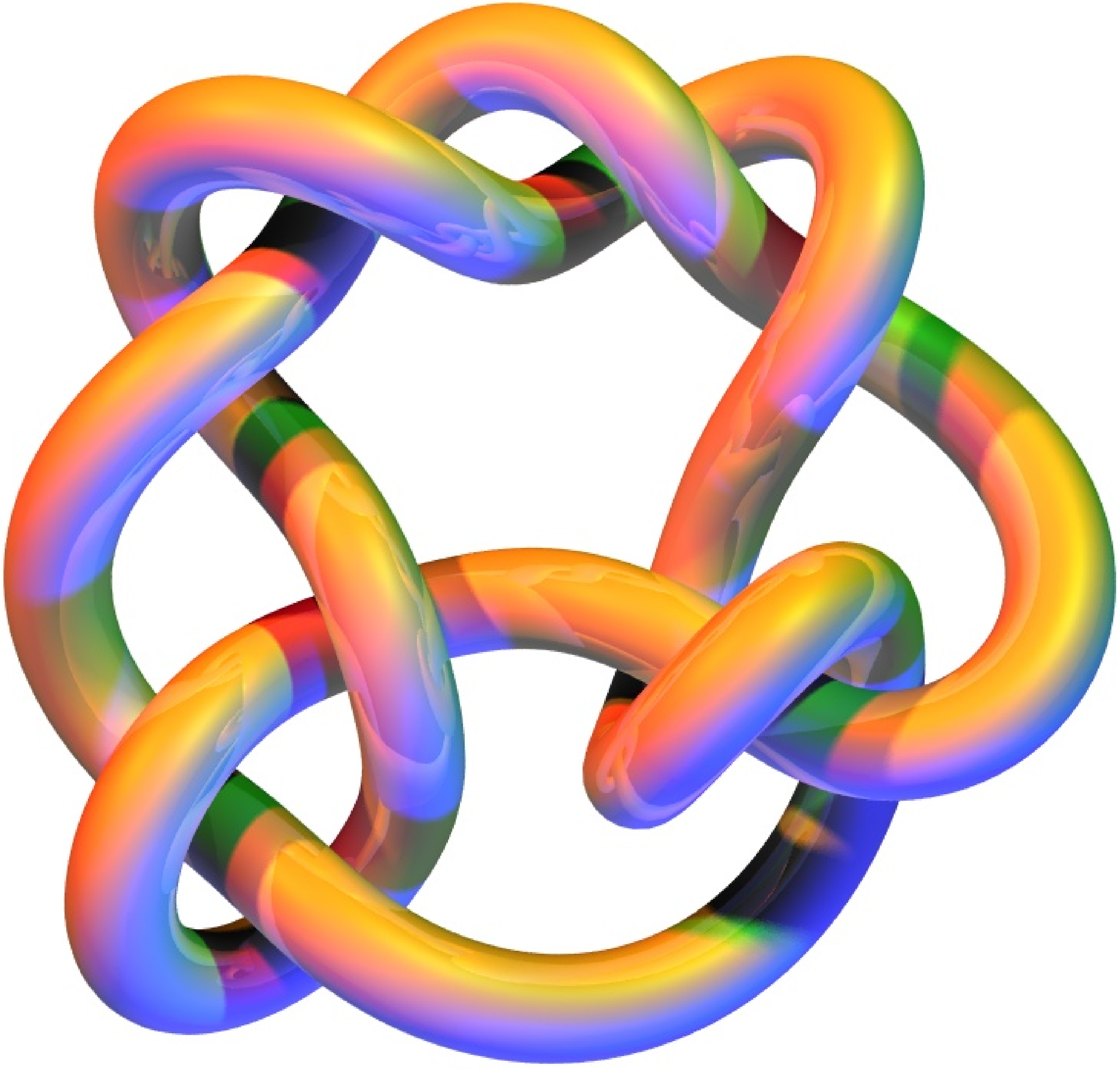
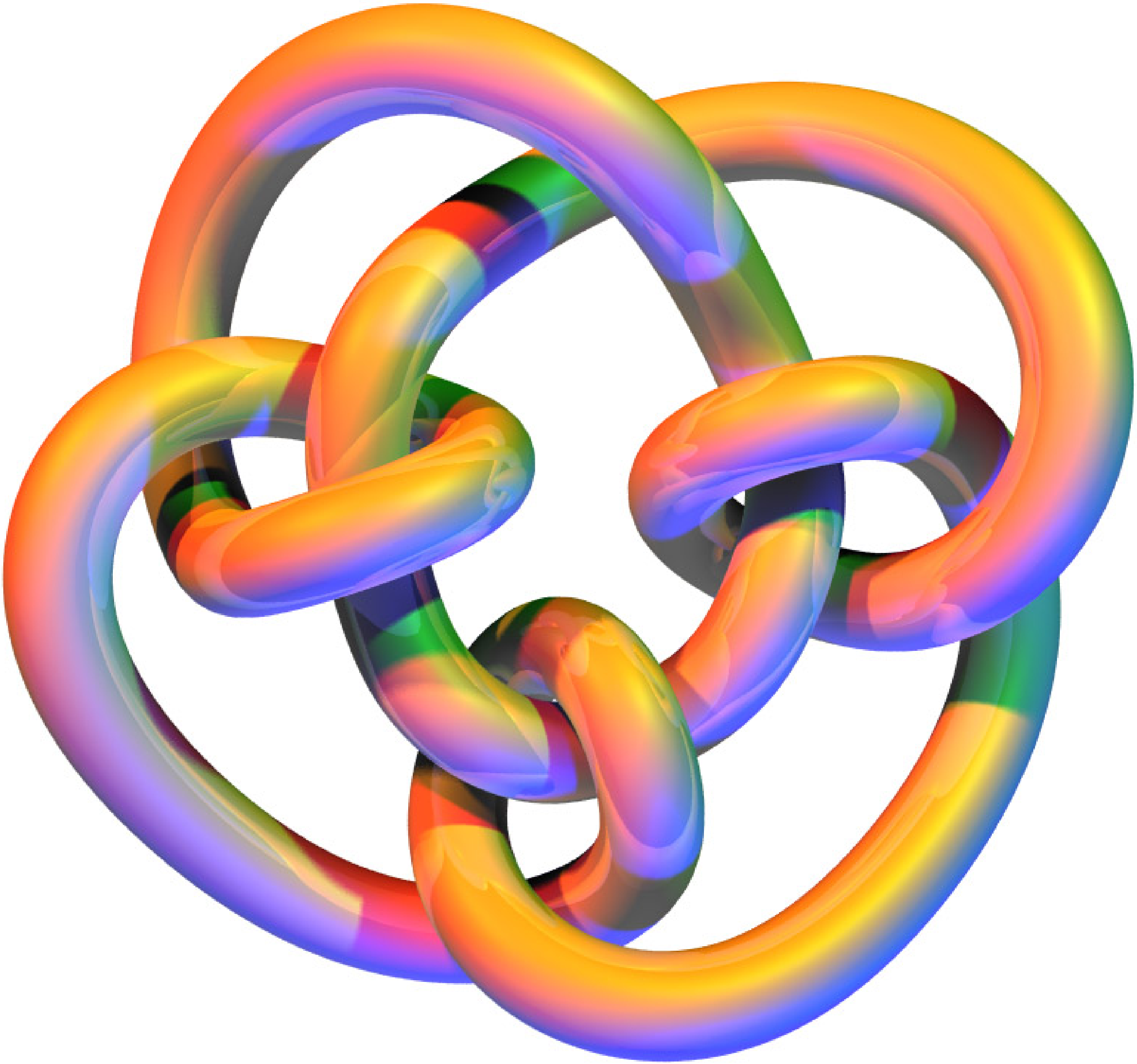
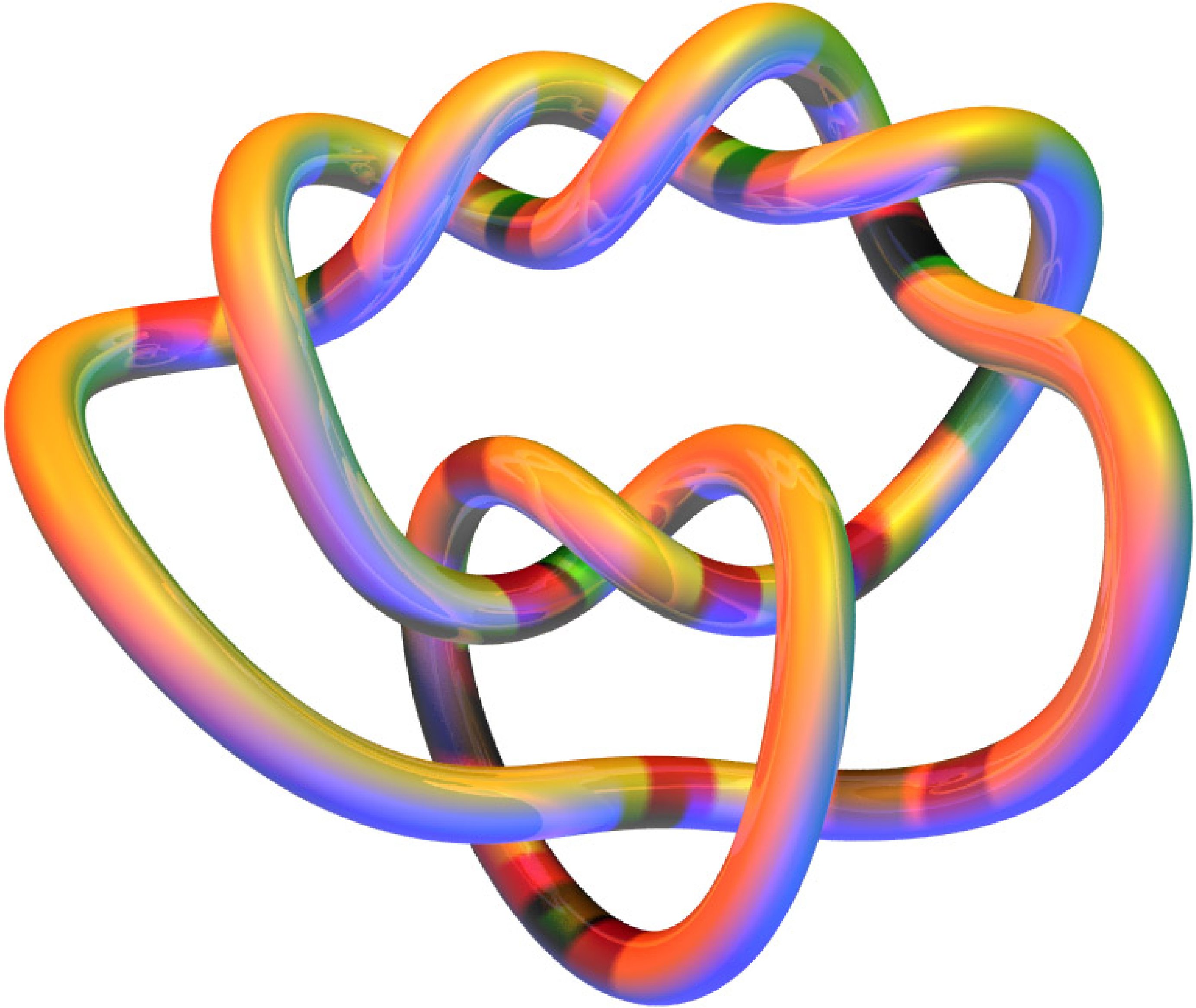
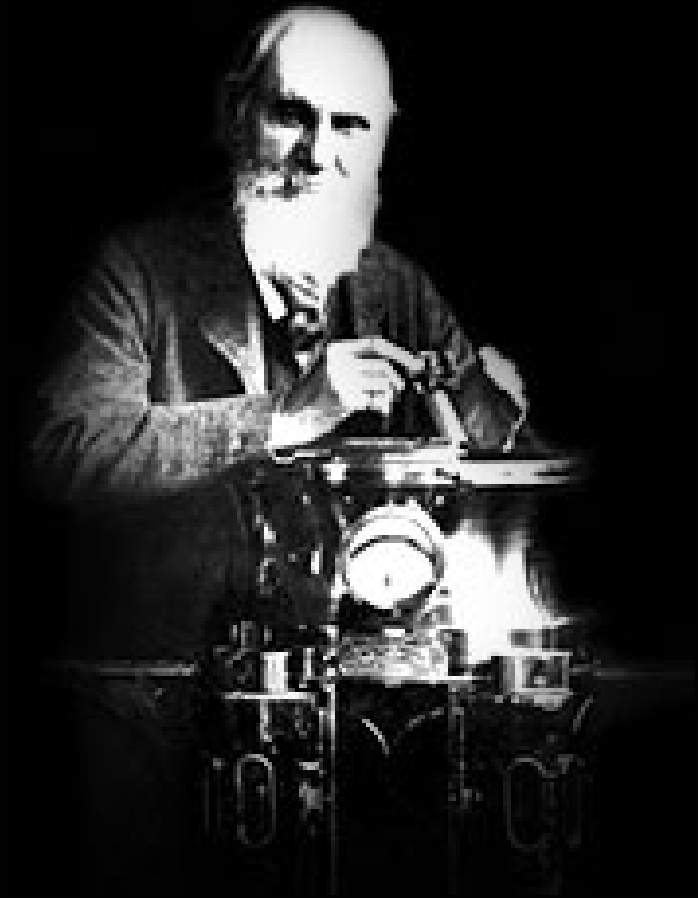




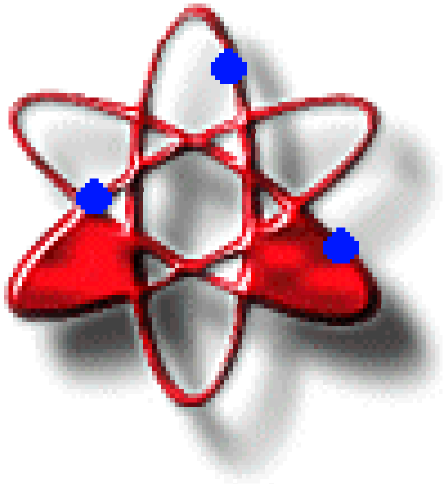




William Thomson
(Lord Kelvin)
1824–1907

Thomson modeled atoms
as knots in the ether

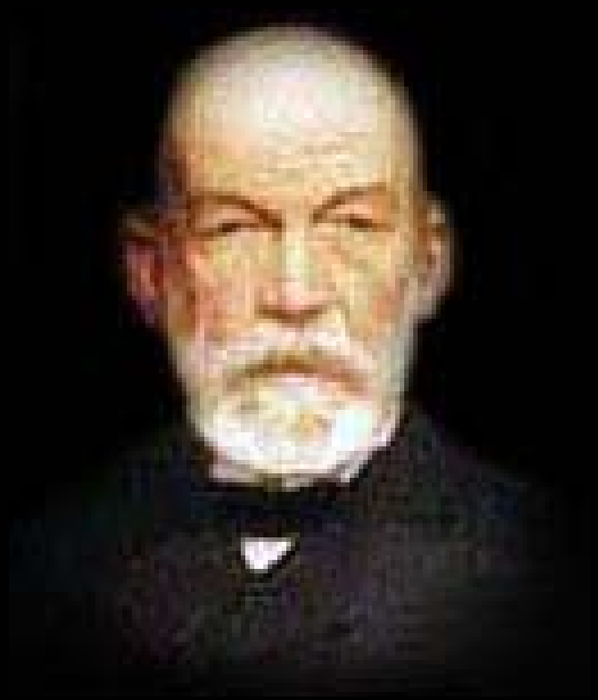
Niels Bohr
1885–1962



Bohr’s model
of the atom

I REALLY THINK KNOTS ARE AS GOOD A DESCRIPTION OF ATOMS AS I’VE SEEN!

James Clerk Maxwell
1831–1879



Peter Guthrie Tait
1831–1901

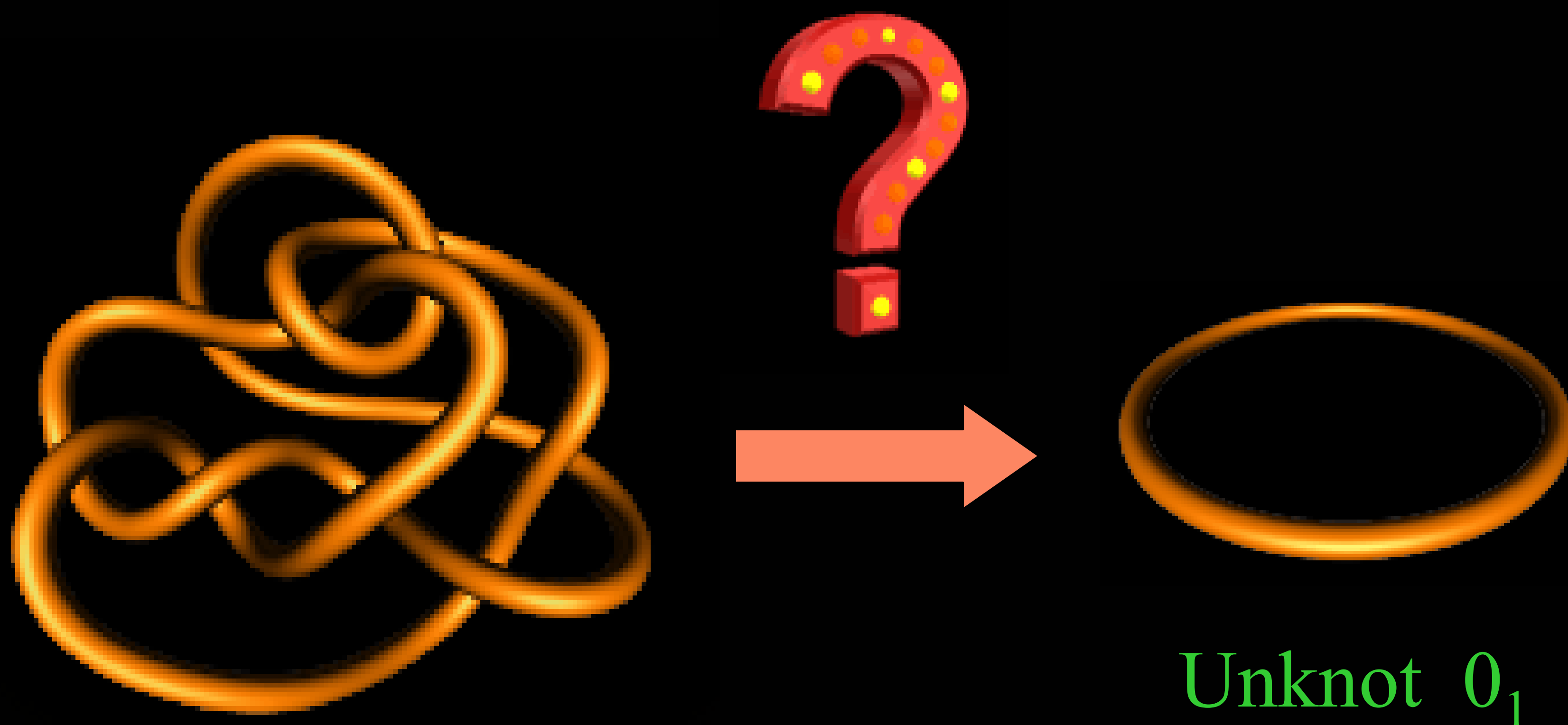
Began to make a catalog of knots…

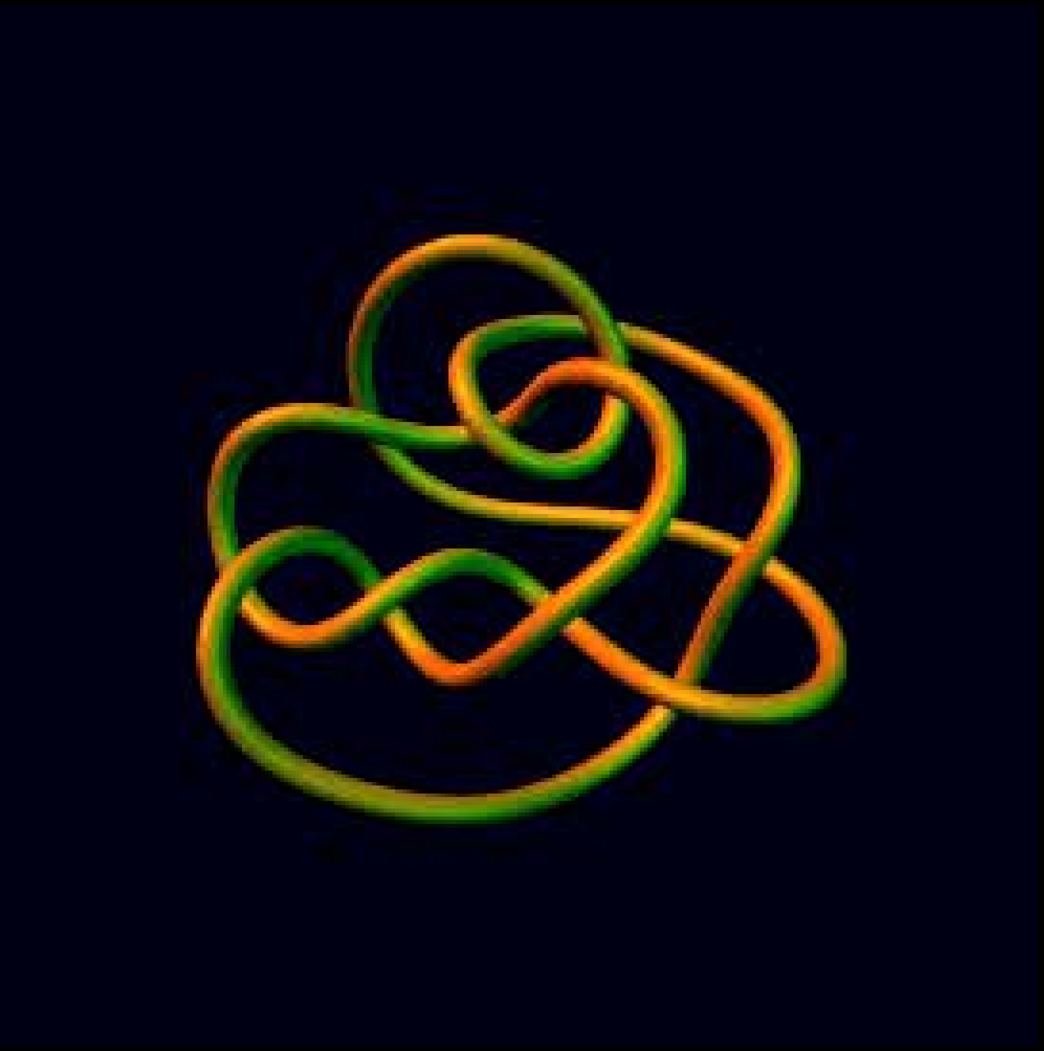
Catalog of Knots

0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1

7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2

etc…


Unknot 0_1



=

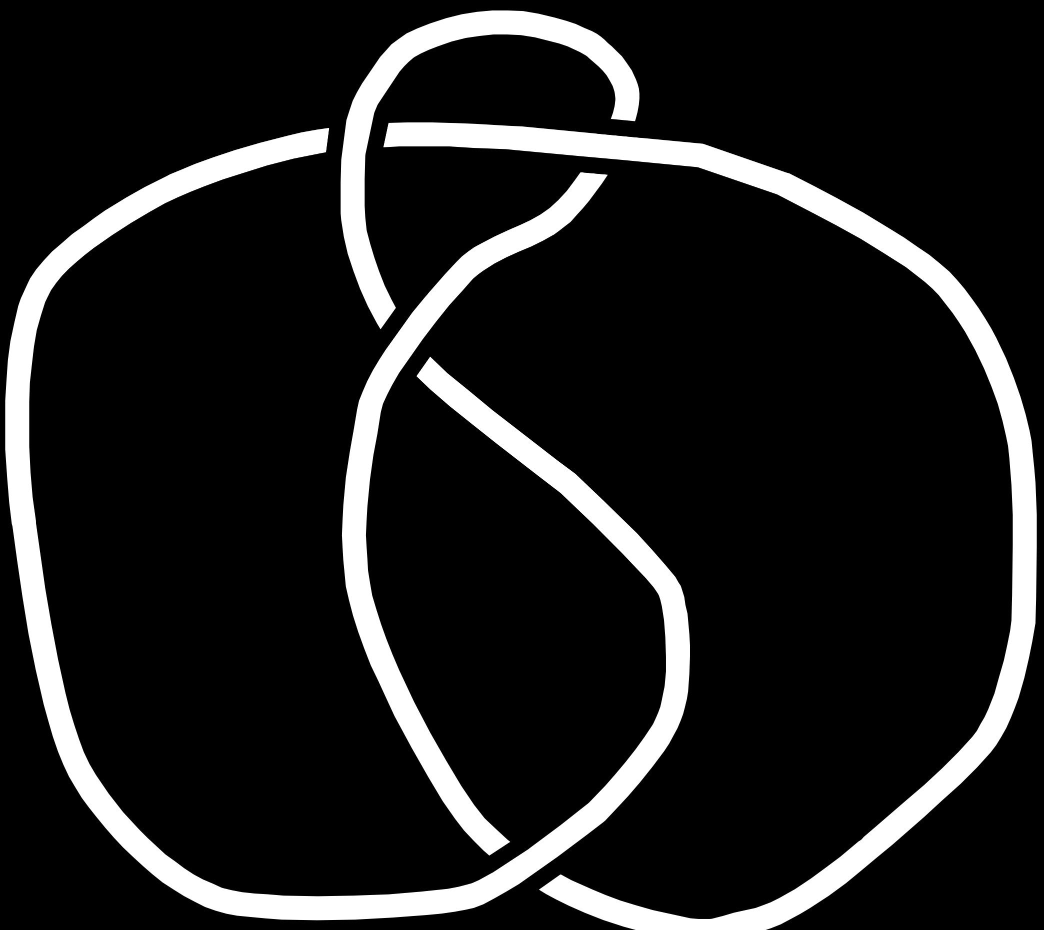
Unknot 0_1

The trefoil knot 3_1 is 3-colorable

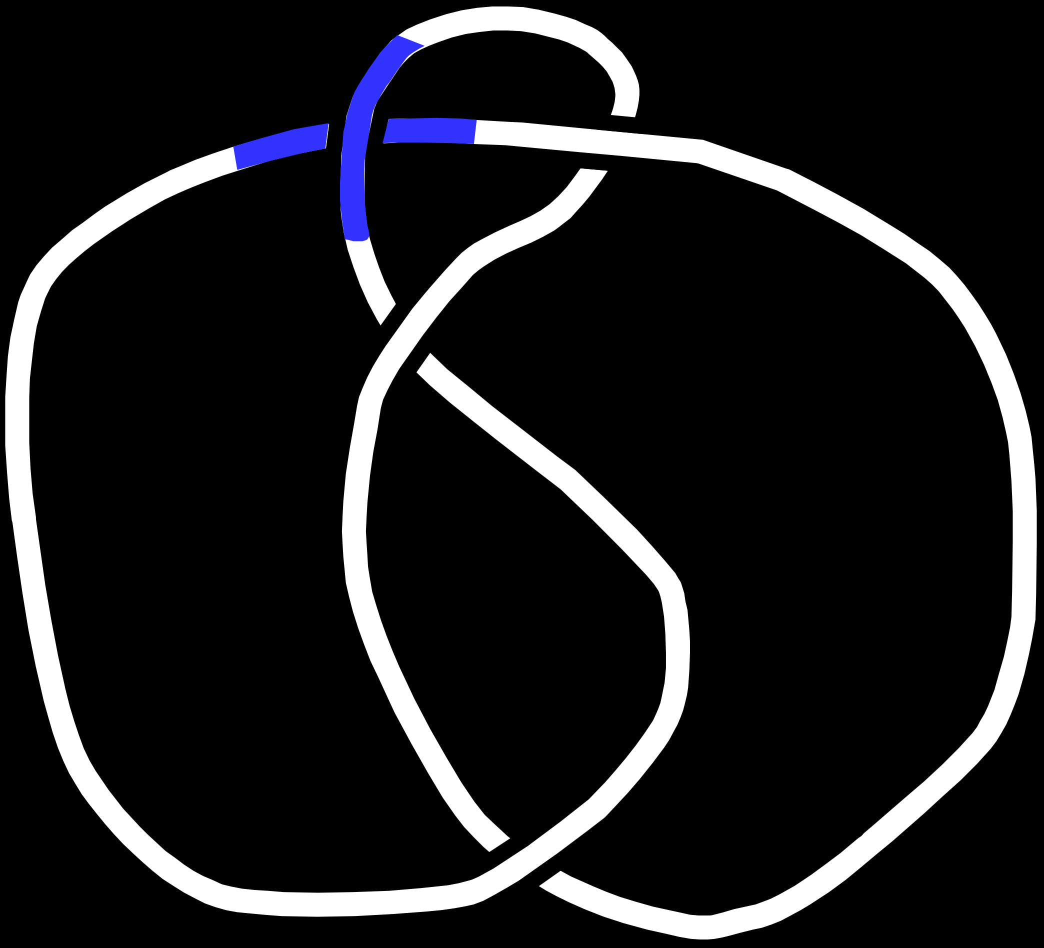
- Arcs of the knot diagram are colored red, green and blue. All three colors must be used.

- At each crossing point, either one color or all three colors appear.

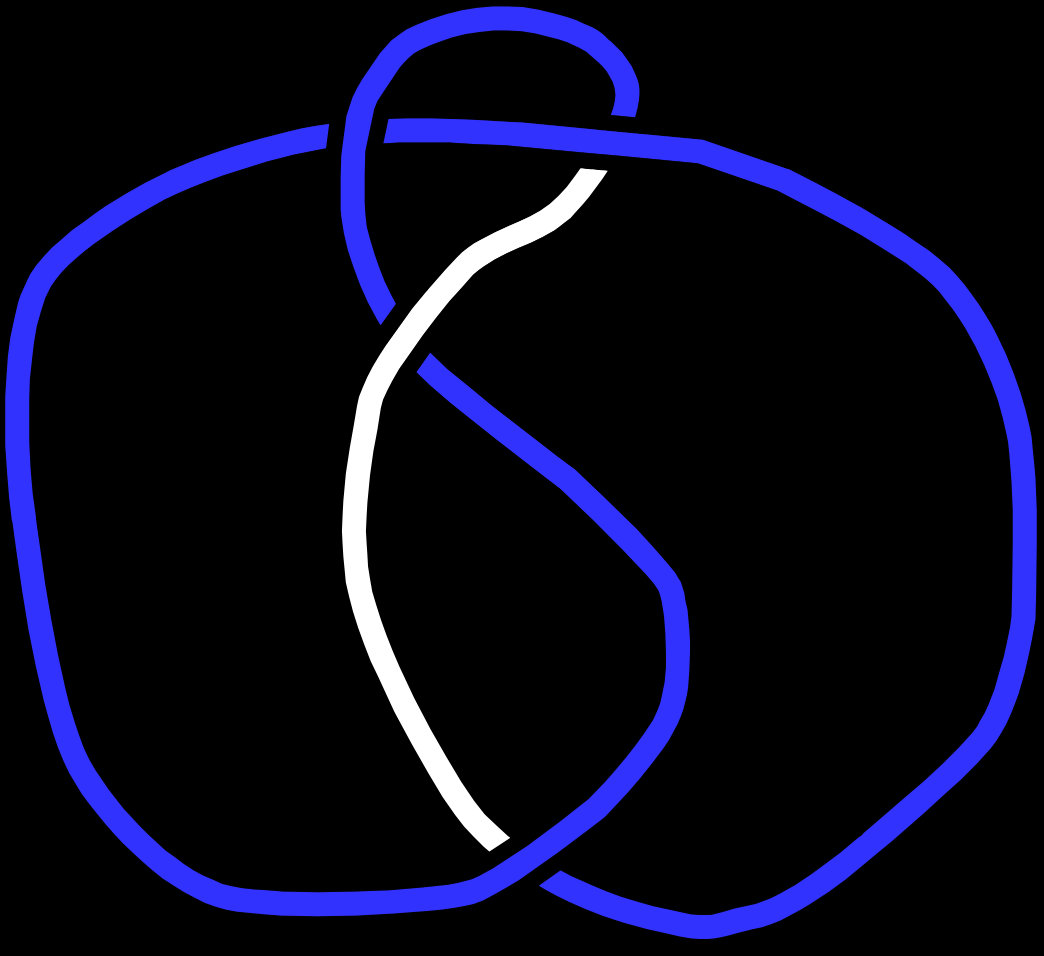
The knot 4_1 is *not* 3-colorable



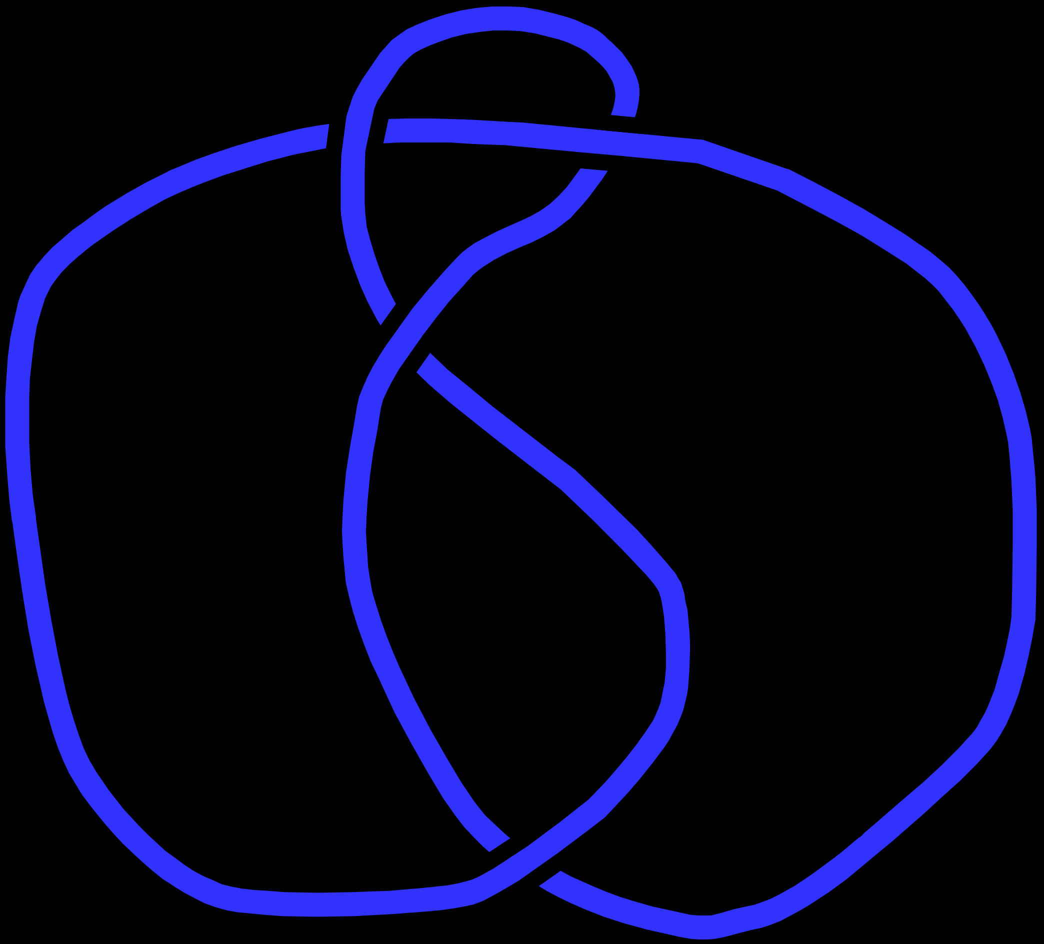
The knot 4_1 is *not* 3-colorable



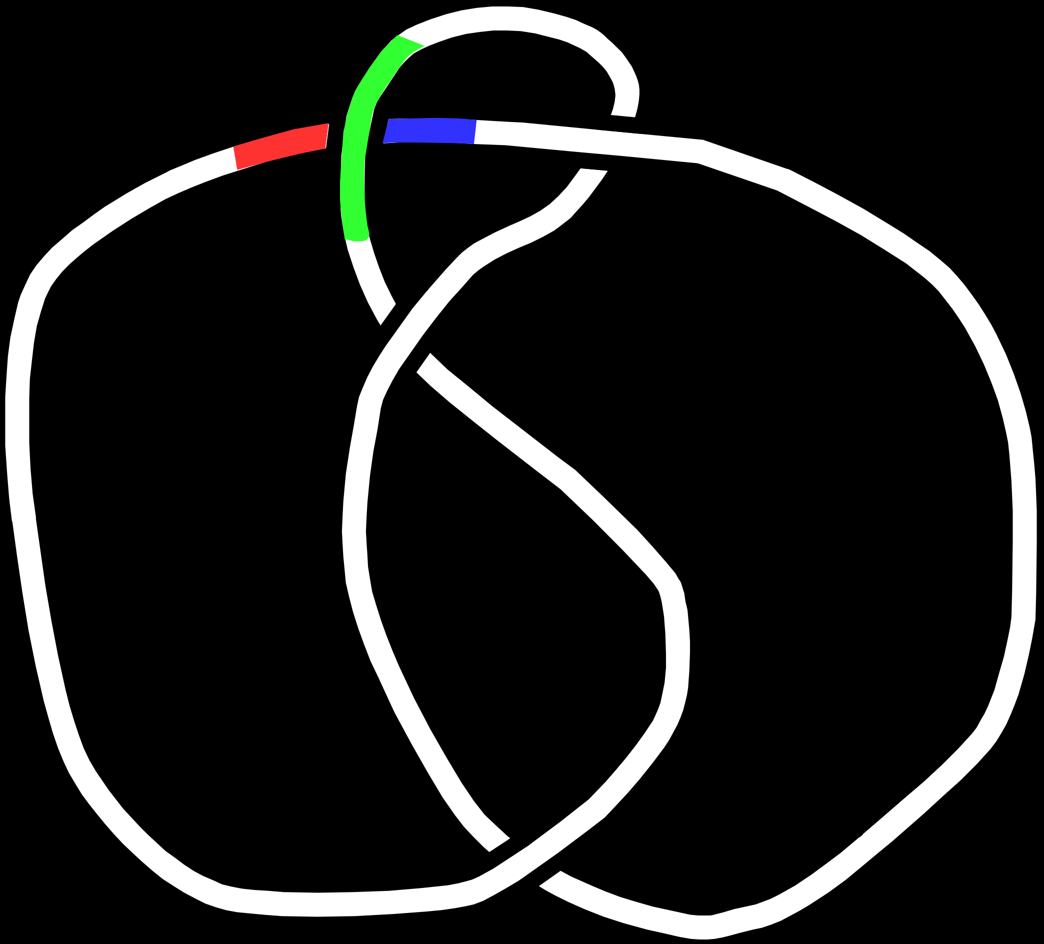
The knot 4_1 is *not* 3-colorable



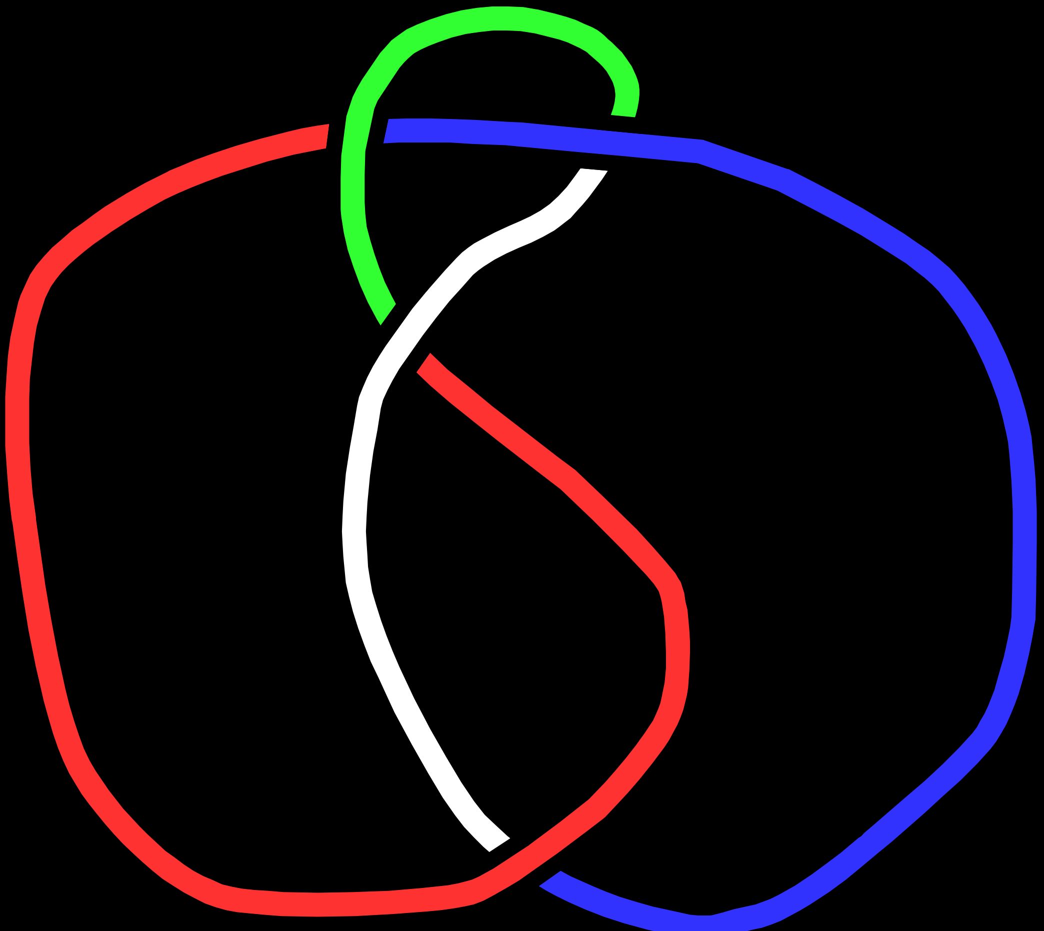
The knot 4_1 is *not* 3-colorable



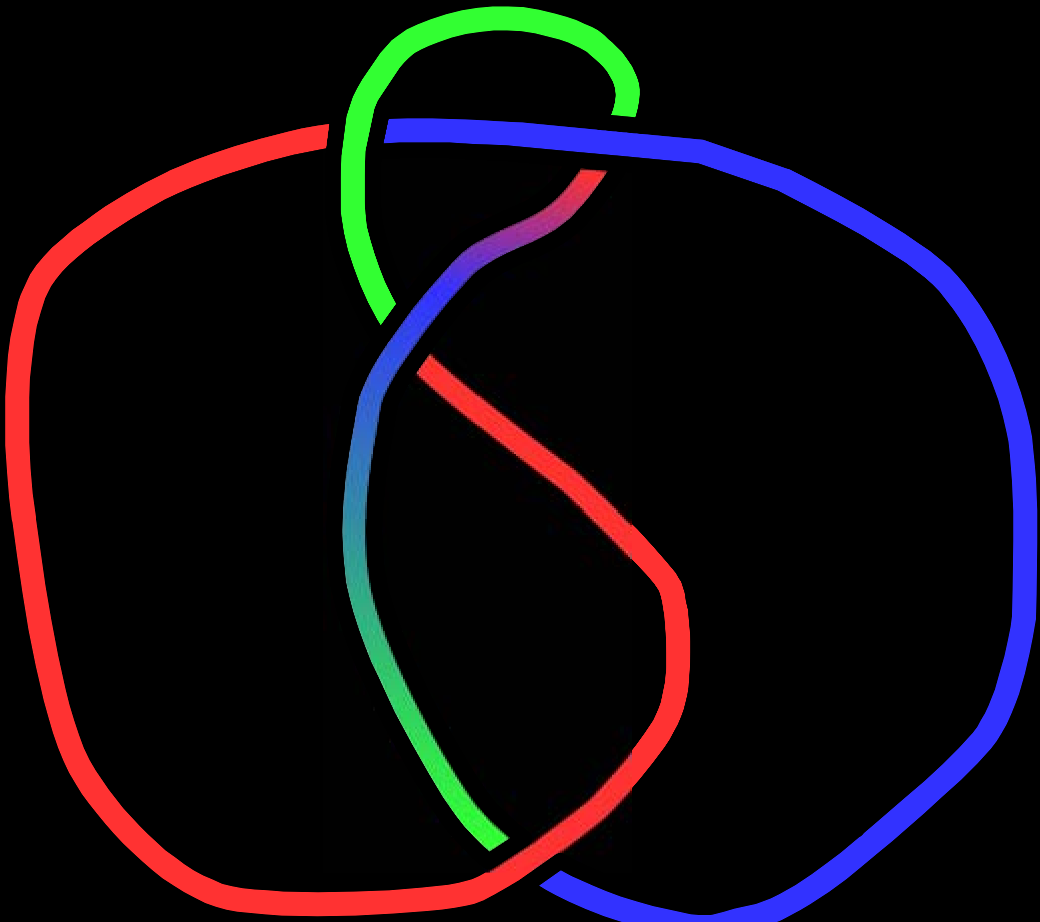
The knot 4_1 is *not* 3-colorable



The knot 4_1 is *not* 3-colorable



The knot 4_1 is *not* 3-colorable



This shows that

3-colorable

Trefoil knot 3_1

$\neq$

not 3-colorable

4_1

And that

3-colorable

Trefoil knot 3_1

$\neq$

not 3-colorable

Unknot 0_1

But why …

not 3-colorable

4_1

$\neq$

not 3-colorable

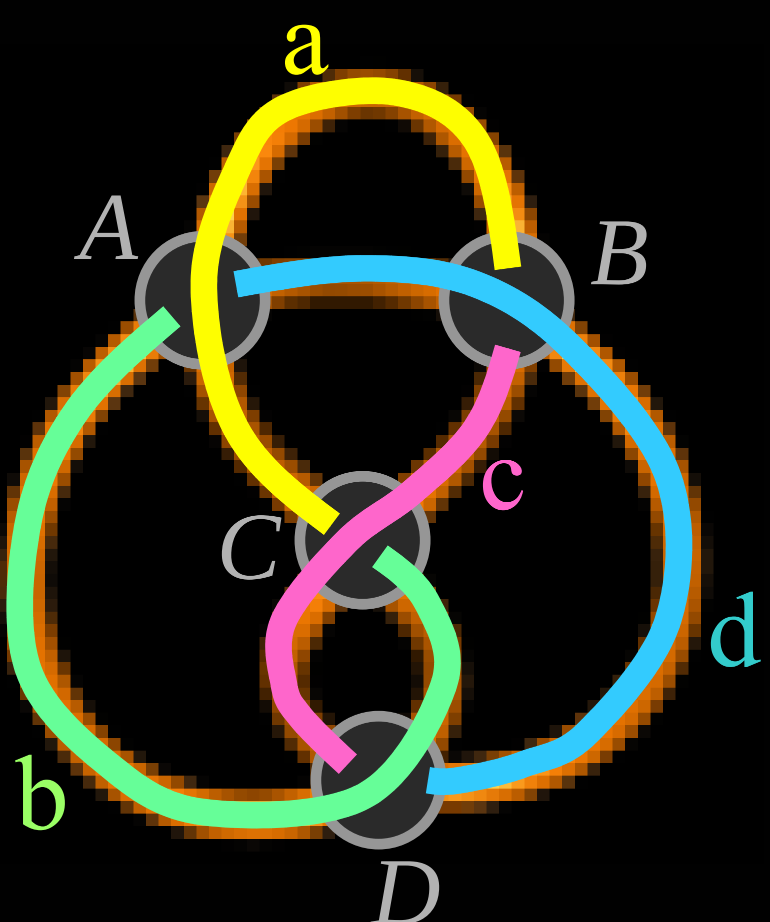
Unknot 0_1

James Alexander
1888–1971

4_1

$$1-3x+x^2$$

Alexander polynomial of the knot 4_1



	a	b	c	d
A	$1-x$	x	x	0
B	-1	0	-1	$1-x$
C	0	-1	$1-x$	-1
D	x	$1-x$	0	x

```
> A:=matrix([[1-x,x,x,0],[-1,0,-1,1-x],[0,-1,1-x,-1],[x,1-
x,0,x]]);
```

$$A := \begin{bmatrix} 1-x & x & x & 0 \\ -1 & 0 & -1 & 1-x \\ 0 & -1 & 1-x & -1 \\ x & 1-x & 0 & x \end{bmatrix}$$

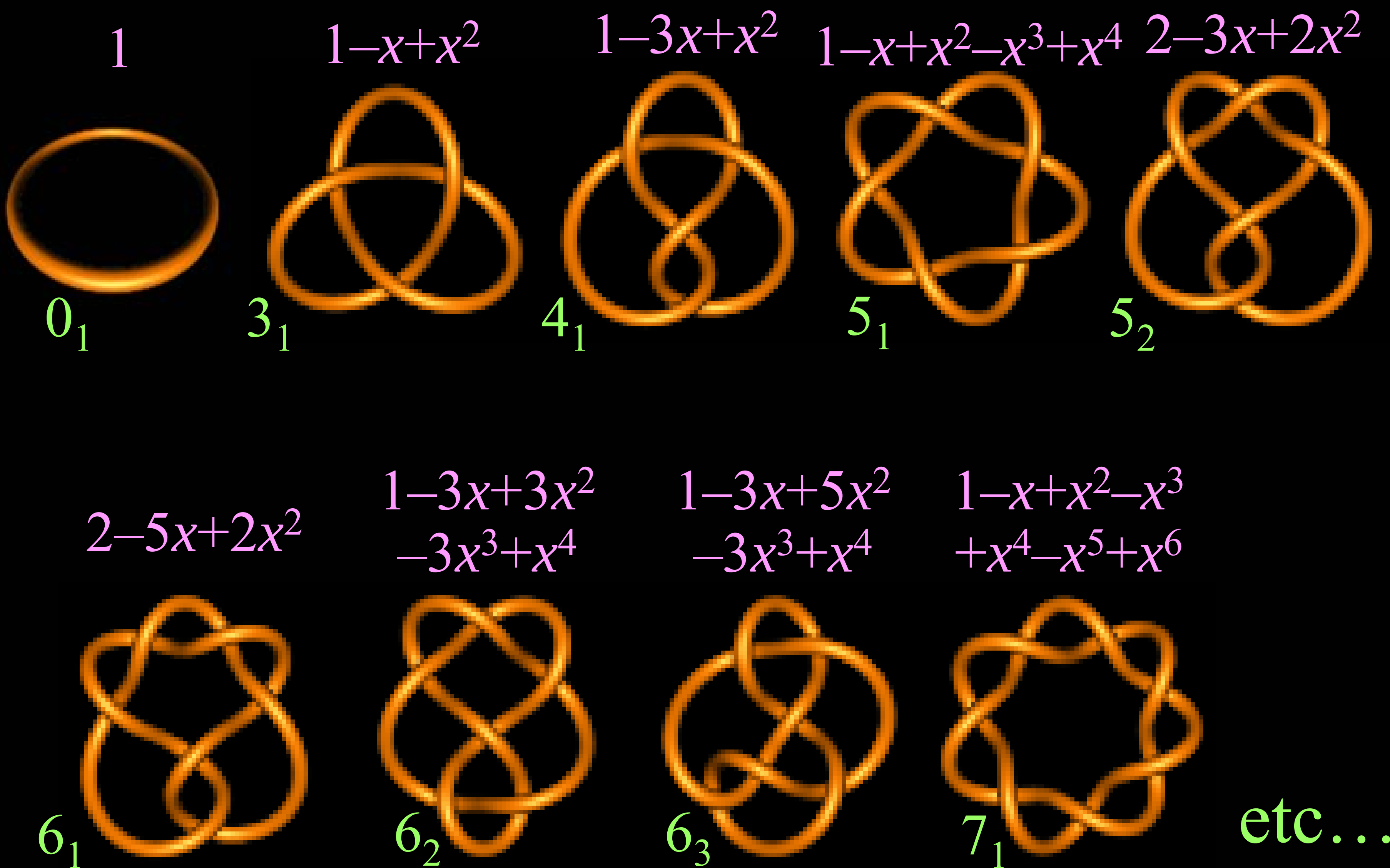
```
> adjoint(A);
```

$$\begin{bmatrix} x^3-3x^2+x & x^3-3x^2+x & x^3-3x^2+x & x^3-3x^2+x \\ x^3-3x^2+x & x^3-3x^2+x & x^3-3x^2+x & x^3-3x^2+x \\ -x^2+3x-1 & -x^2+3x-1 & -x^2+3x-1 & -x^2+3x-1 \\ -x^2+3x-1 & -x^2+3x-1 & -x^2+3x-1 & -x^2+3x-1 \end{bmatrix}$$

$$1-3x+x^2$$

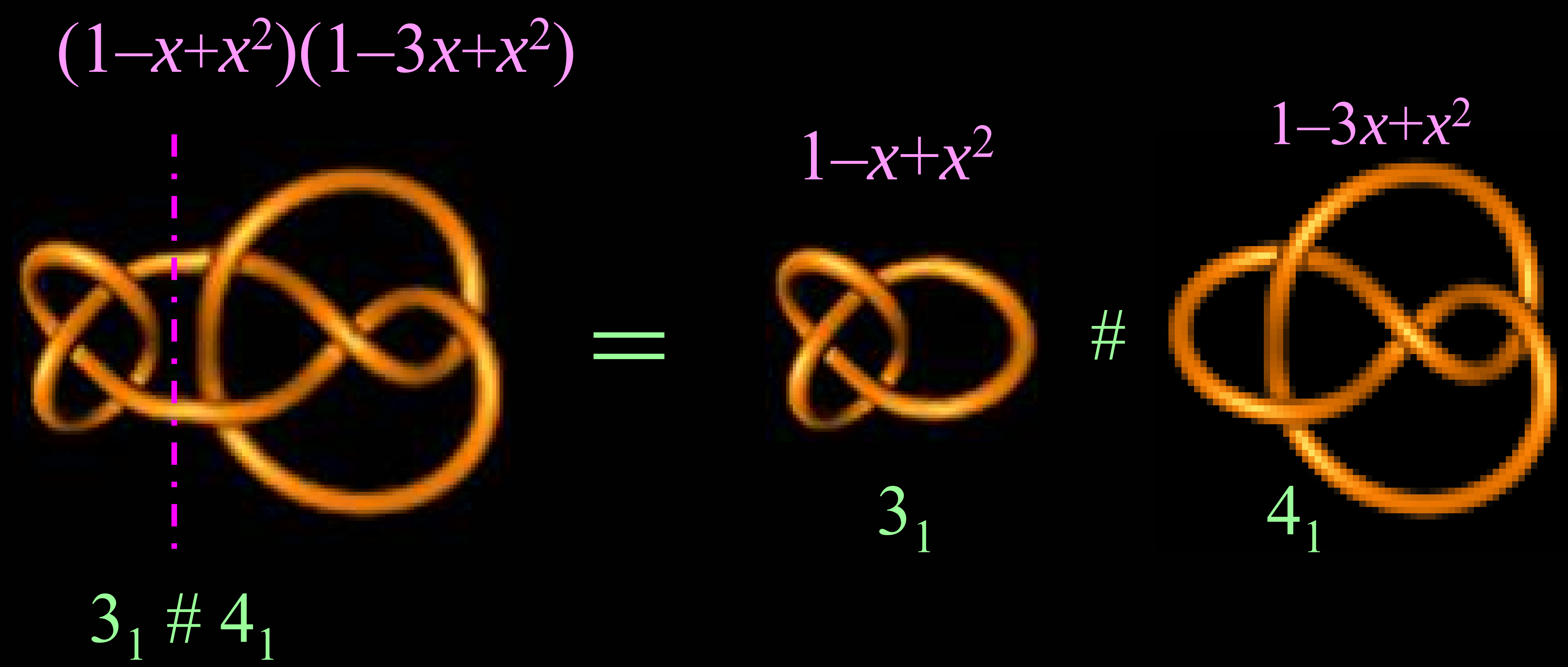
Alexander polynomial of the knot 4_1

Alexander Polynomials of Knots



Prime Factorization of Knots

and their Alexander polynomials

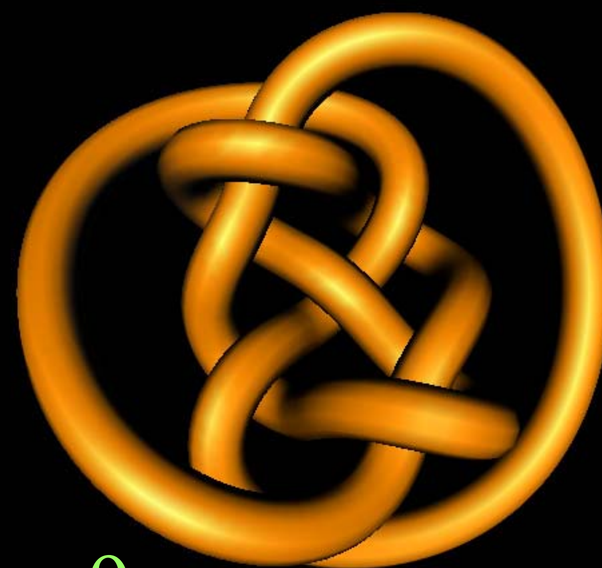


But inequivalent knots sometimes have the same Alexander polynomial!

$1-5x+12x^2-15x^3+12x^4-5x^5+x^6$

9_{28}

$1-5x+12x^2-15x^3+12x^4-5x^5+x^6$



9_{29}

Found connections between von Neumann algebras and geometric topology, resulting in a new polynomial invariant for knots.

Vaughn Jones
(1952–   )

Awarded the Fields Medal in 1990

Jones Polynomials of Knots

1

0_1

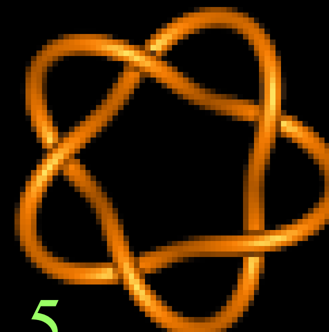
$-x^{-4}+x^{-3}+x^{-1}$

3_1

$x^{-2}-x^{-1}+1$
$-x+x^{2}$

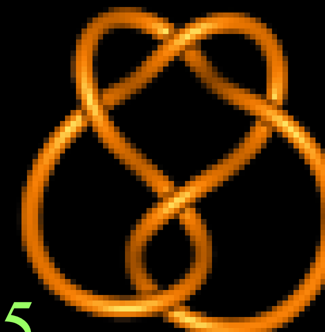
4_1

$-x^{-7}+x^{-6}-x^{-5}$
$+x^{-4}+x^{-2}$



5_1

$-x^{-6}+x^{-5}-x^{-4}$
$+2x^{-3}-x^{-2}+x^{-1}$



5_2

$x^{-4}-x^{-3}+x^{-2}$
$-2x^{-1}+2-x+x^{2}$

6_1

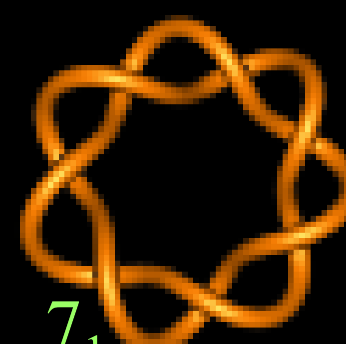
$x^{-5}-2x^{-4}+2x^{-3}$
$-2x^{-2}+2x$
$-1+x$

6_2

$-x^{-3}+2x^{-2}$
$-2x^{-1}+3$
$-2x+2x^{2}-x^{3}$

6_3

$-x^{-10}+x^{-9}$
$-x^{-8}+x^{-7}$
$-x^{-6}+x^{-5}+x^{3}$



7_1

etc…

These inequivalent knots have *different* Jones polynomials!

$$-x^{-2}+3x^{-1}-5$$
$$+8x-8x^{2}+9x^{3}-8x^{4}$$
$$+5x^{5}-3x^{6}+x^{7}$$

$$-x^{-6}+3x^{-5}-6x^{-4}$$
$$+8x^{-3}-8x^{-2}+9x^{-1}-7$$
$$+5x-3x^{2}+x^{3}$$

9_{28}

9_{29}

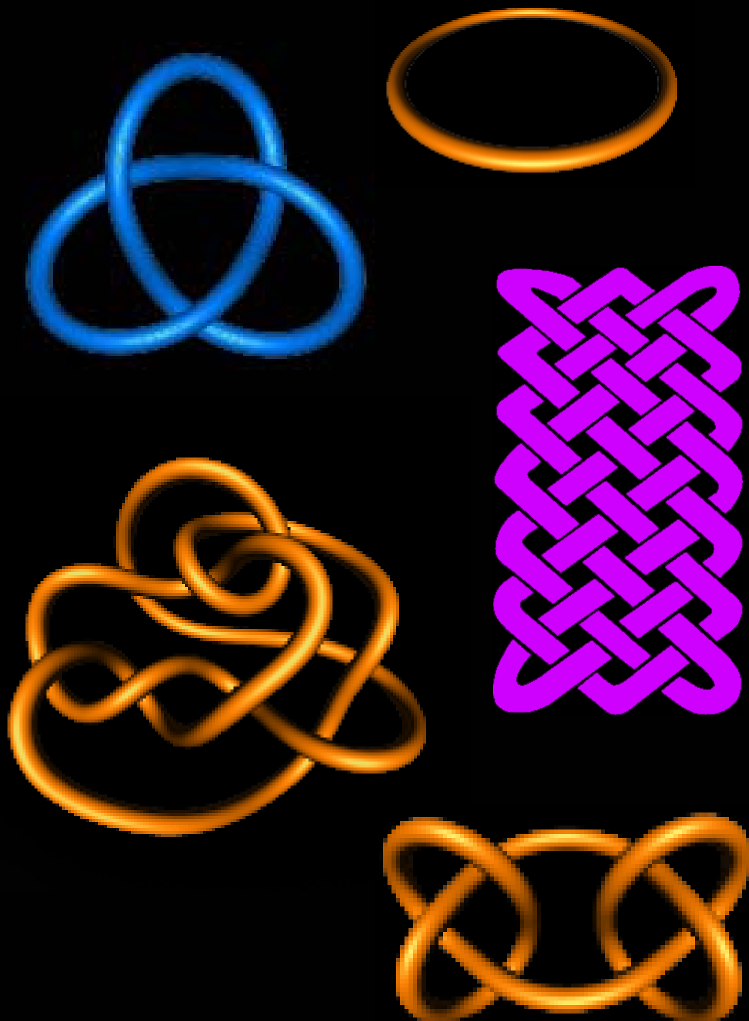
There *do exist* inequivalent knots with the same Jones polynomial.

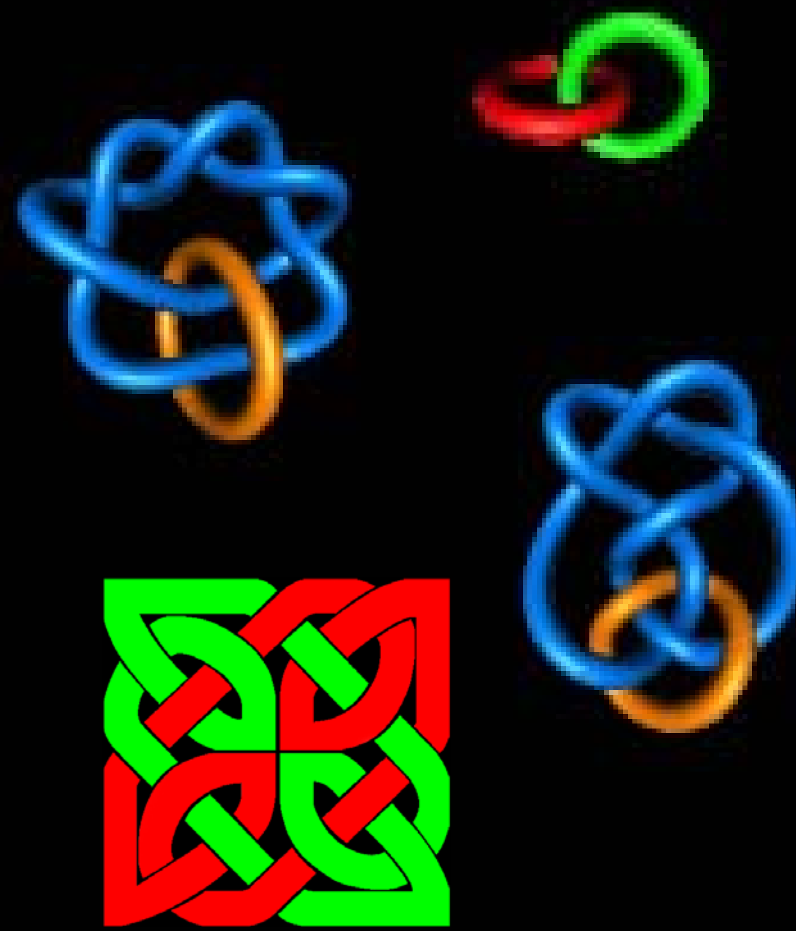
It is *not known* whether the unknot is the only knot with Jones polynomial = 1.

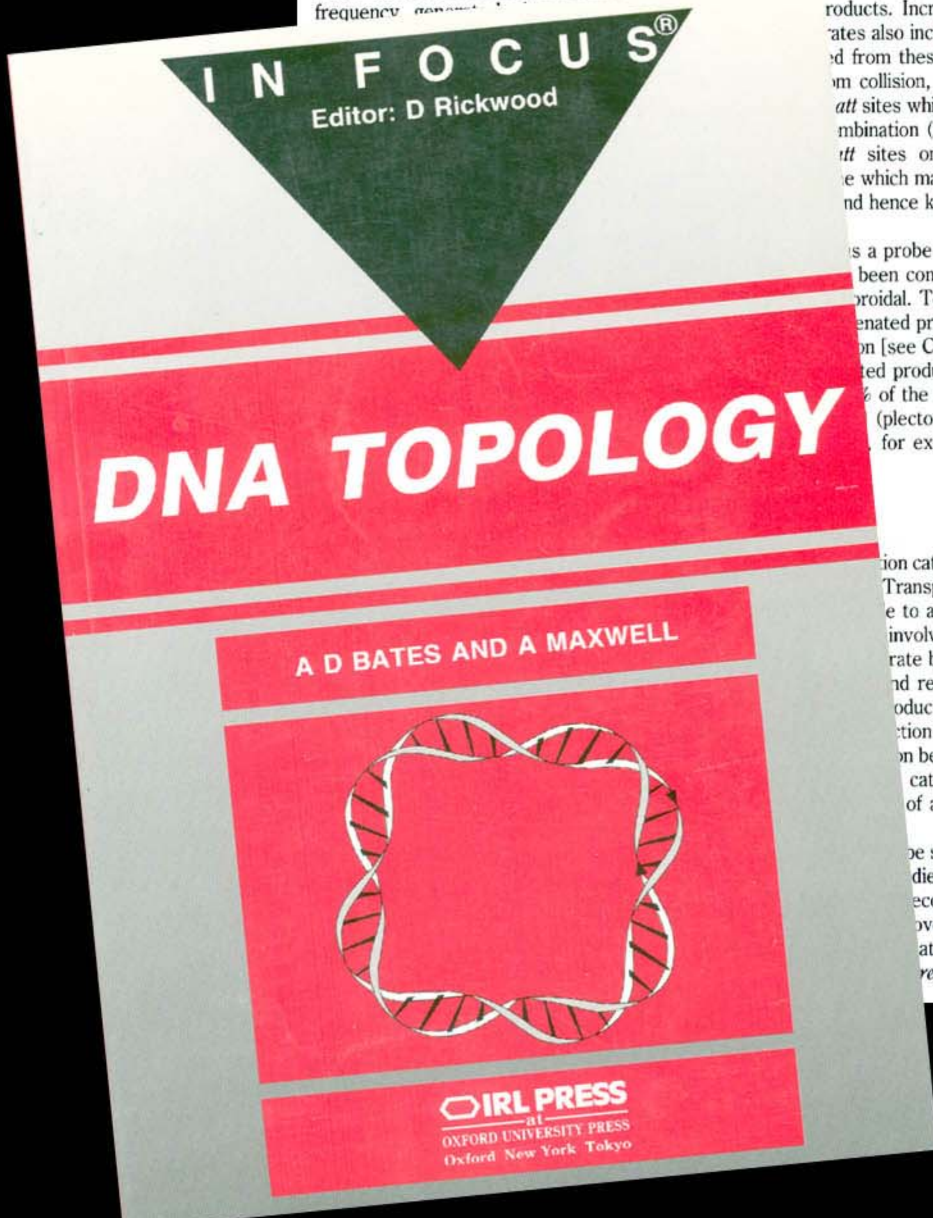
Unknot 0_1

Jones polynomial = 1

Knots vs. Links







linking difference of the substrate. This is due to the trapping of writhes (or nodes) in interwound DNA (see Chapter 2, Section 4.2) between the interacting recombination sites. Non-supercoiled (nicked-circular) DNA molecules can, under certain conditions, constitute substrates f... ination and, at low frequency, gen... roducts. Increasing
...ates also increases
...d from these (and
...m collision, which
att sites which will
...mbination (*Figure*
tt sites on non-
...e which may lead
...nd hence knotted

...s a probe of the
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...oroidal. Toroidal
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...n [see Chapter
...ted products of
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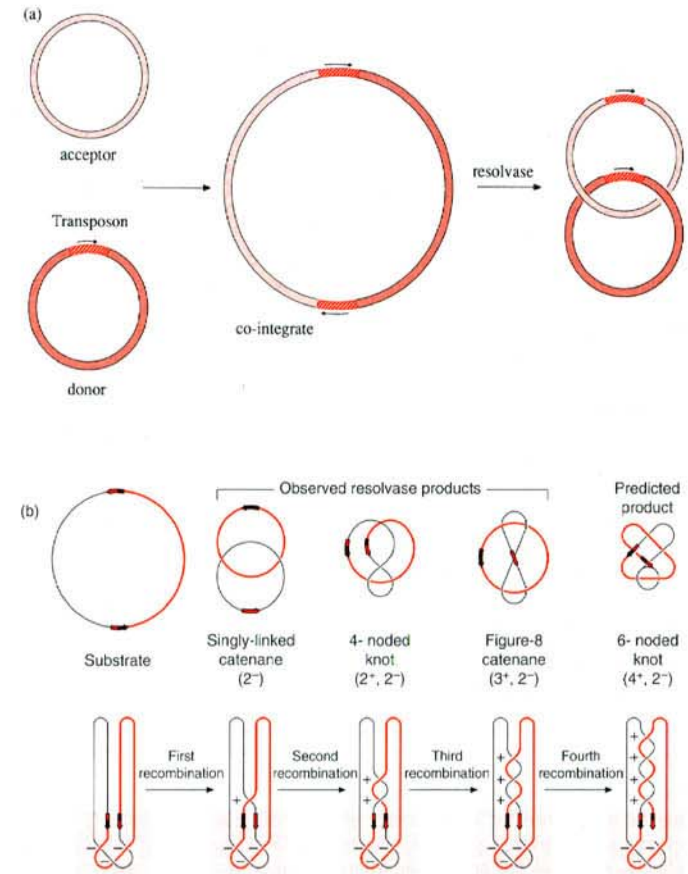


Figure 6.7. Site-specific recombination by resolvase. **(a)** A donor circle carrying a transposon of the Tn3 family forms a co-integrate with the acceptor DNA circle. The co-integrate carries two copies of the transposon. The action of resolvase catalyses a recombination reaction between *res* sites within the two transposons to yield the catenated product circles, each carrying a single copy of the transposon. **(b)** Scheme for the formation of multiple knotted and catenated products by Tn3 resolvase. Successive rounds of recombination generate the products shown in the upper row. These products can be rationalized by proposing a synaptic process involving three nodes (lower row) (b redrawn from ref. 84. Copyright 1985 by the AAAS).

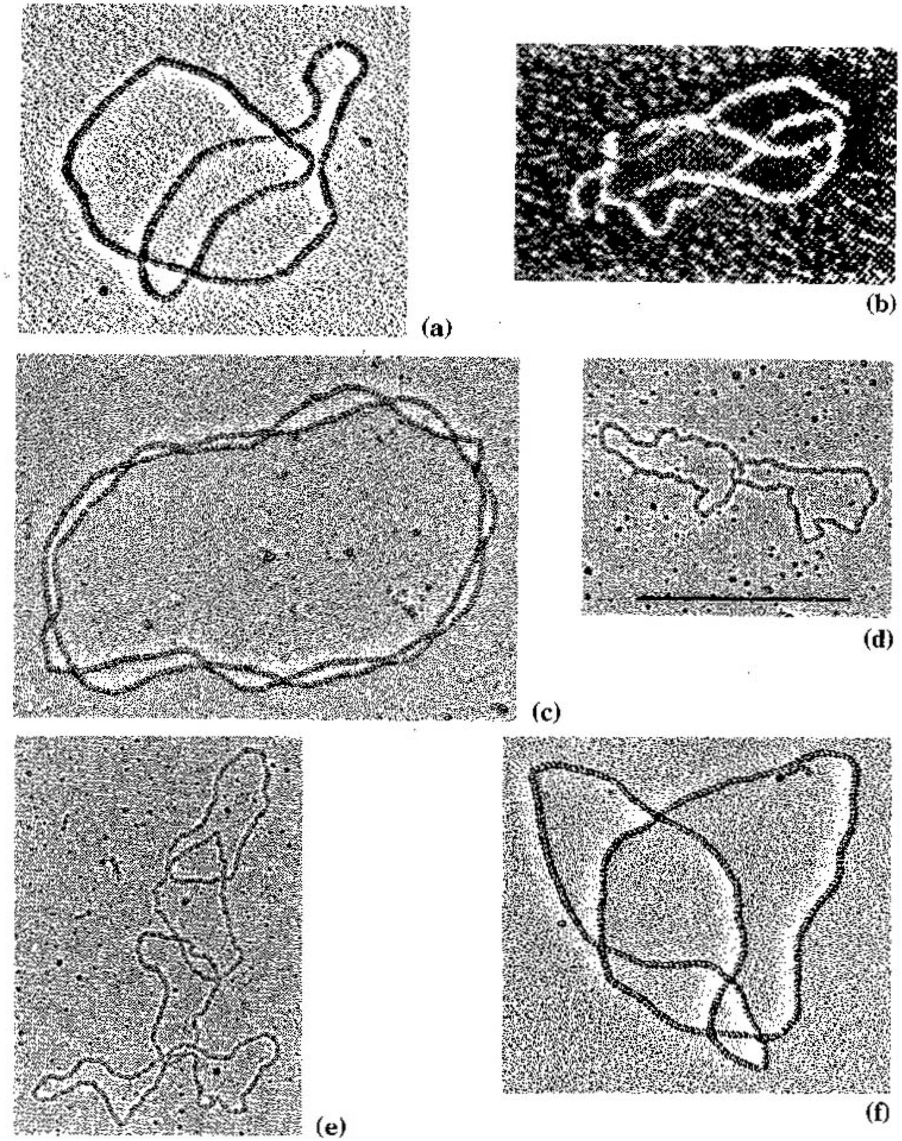
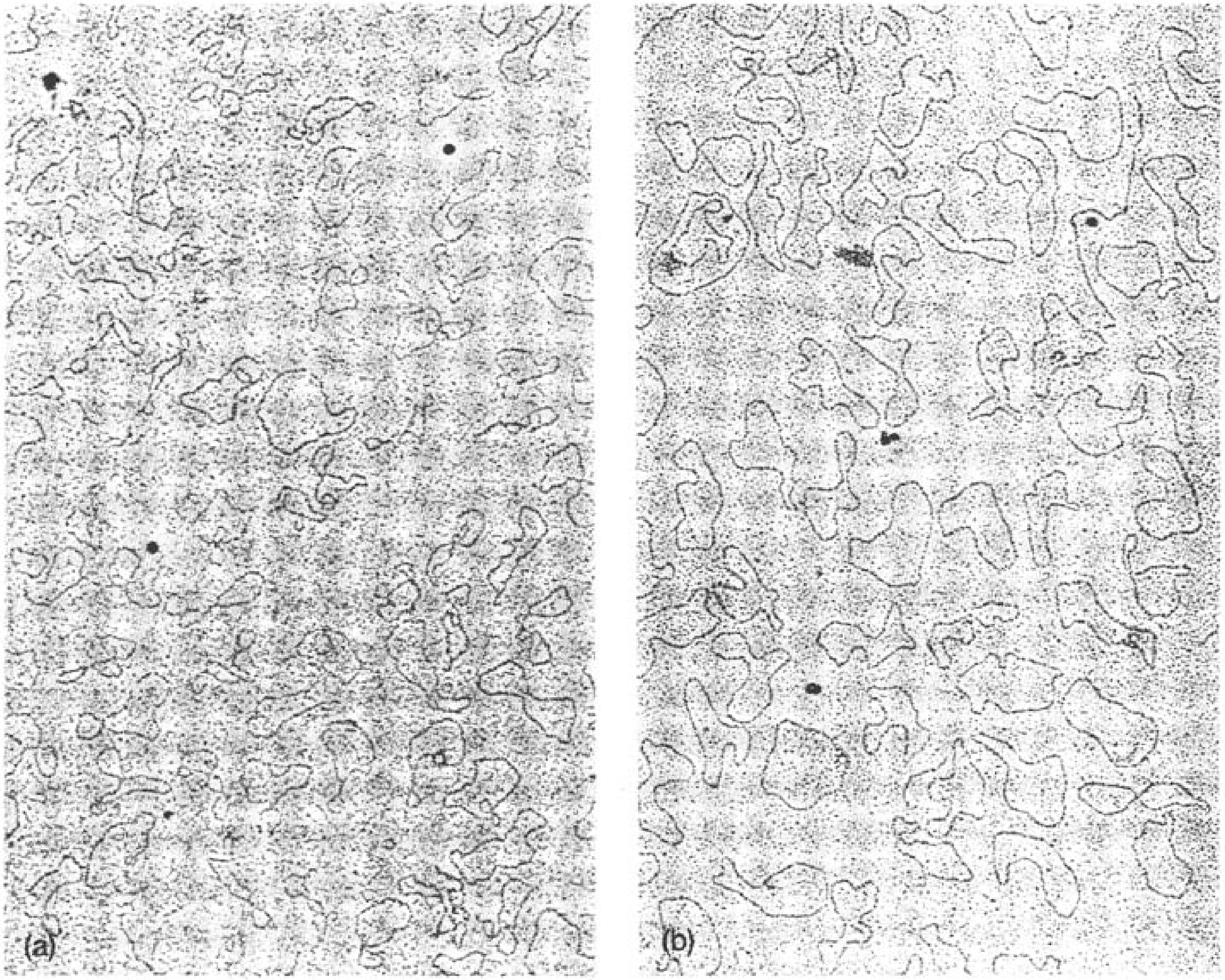
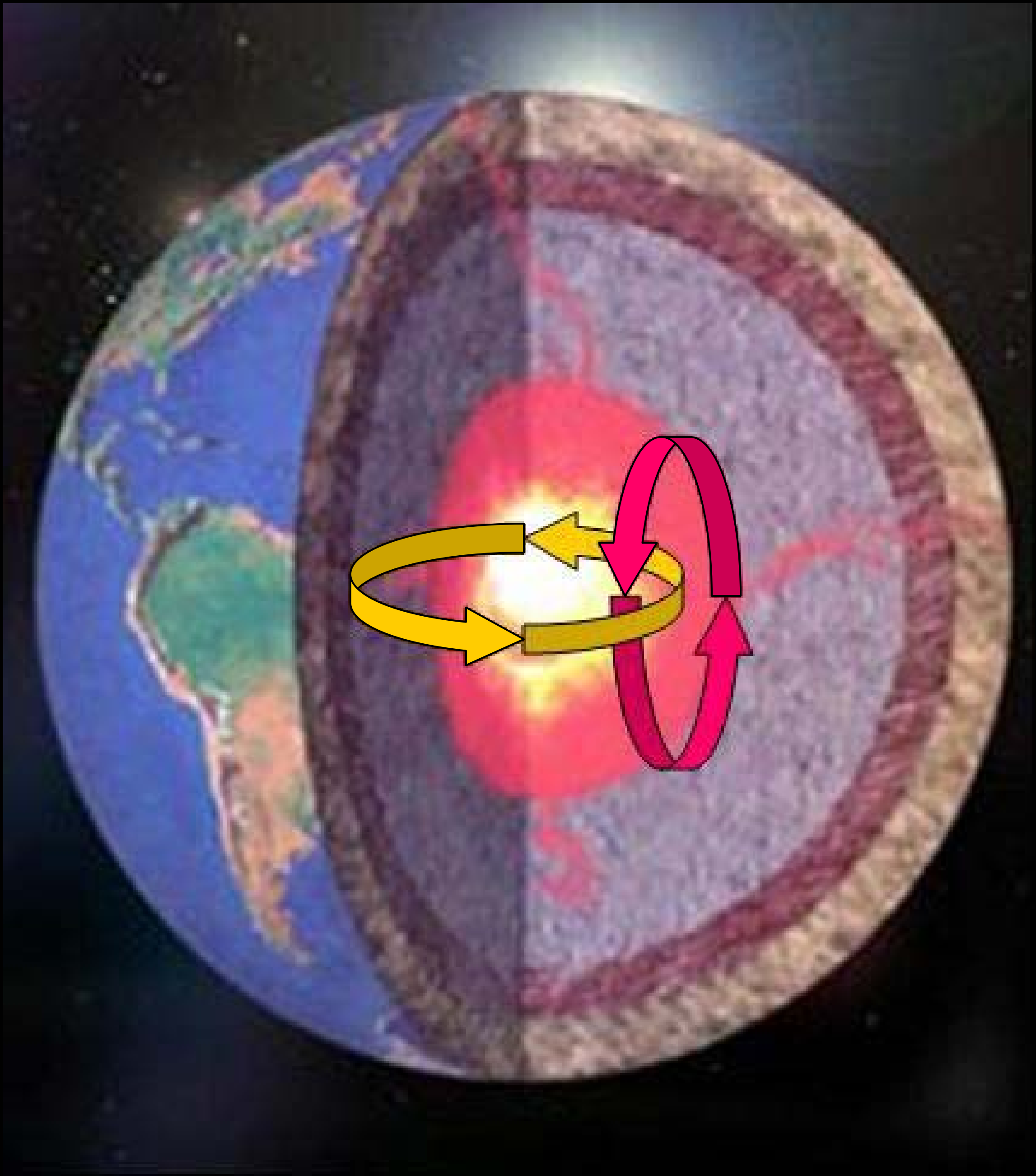
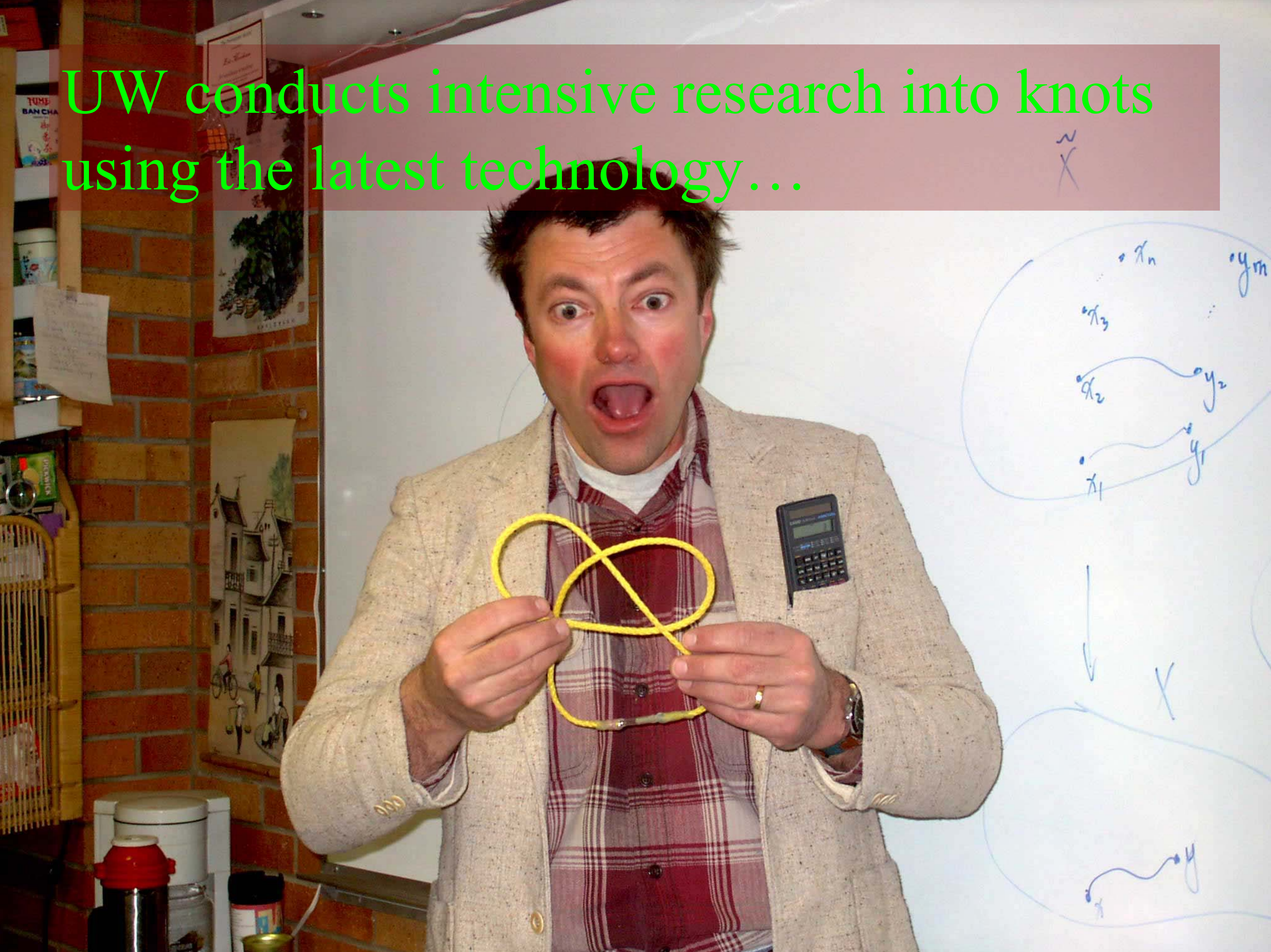


Figure 4.1. Electron micrographs of DNA knots and catenanes. Knotted and catenated DNA were coated with RecA protein prior to visualization by electron microscopy. **(a)** Trefoil (3-noded knot) (reproduced from ref. 6). **(b)** Five-noded knot (reproduced from ref. 26). **(c)** 13-noded torus knot (reproduced from ref. 27). **(d)** Singly-linked catenane (reproduced from ref. 23). **(e)** Catenane consisting of five circles (reproduced from ref. 23). **(f)** Figure-eight (five-noded) catenane (reproduced from ref. 6). (a), (f) reprinted by permission; Copyright © 1983 Macmillan Magazines Limited; (c), (d), (e) Copyright 1980, 1985 Cell Press.


(a)
(b)




UW conducts intensive research into knots
using the latest technology…

www.pims.math.ca/KnotPlot/

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