The 2-transitive complex Hadamard matrices

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A *complex Hadamard matrix* is a $v \times v$ matrix H whose entries are complex roots of unity, such that $HH^* = vI$. ($H^* =$ conjugate-transpose of H)

Ordinary Hadamard matrix: entries ± 1

Butson (1963) for p-th roots of 1, p prime

Turyn (1970): entries ± 1, $\pm i$

Example 1:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \quad \omega = e^{2\pi i/3}.$$

More generally, the character table of any finite abelian group.

Example 2:

$$H_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \end{bmatrix}, \quad \omega = e^{2\pi i/3}.$$

$H_6 \longleftrightarrow$ antipodal distance-regular graph of diameter 4 on 36 vertices (a triple cover of the complete bipartite graph $K_{6,6}$)

The Problem

Determine, to within 'equivalence', all complex Hadamard matrices H having an 'automorphism group' 2-transitive on the rows.

An *automorphism* of H is a pair (M_1, M_2) of monomial matrices (the nonzero entries of M_i are complex roots of 1, one per row/column) such that $M_1 H M_2^* = H$.

Let G be a finite group of automorphisms of H.

Suppose entries of H are p-th roots of unity, p prime. Then

$H \longleftrightarrow$ antipodal distance-regular graph Γ (a p-fold cover of $K_{v,v}$) with an automorphism of order p fixing every fibre.

$$\begin{array}{r} \Gamma \text{ distance-transitive } \Longrightarrow \\ \Gamma \text{ vertex-transitive } \not\Longleftarrow \end{array} \begin{cases} G \text{ 2-transitive} \\ \text{on rows of } H \end{cases}$$

Ivanov, Liebler, Penttila and Praeger (1997): Classified antipodal distance-transitive covers of $K_{v,v}$

Return to **Example:**

$$H_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega & 1 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^2 & \omega & 1 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 \end{bmatrix}, \quad \omega = e^{2\pi i/3}.$$

$H_6 \longleftrightarrow \Gamma$, an antipodal distance-transitive of diameter 4 on 36 vertices.

The cover $\Gamma \rightarrow K_{6,6}$ (complete bipartite graph on $6+6$ vertices) has fibre size 3.

$Aut(\Gamma) \cong 3Aut(Sym_6)$.

H_6 admits $G \cong 3Alt_6$ permuting the rows (and columns) 2-transitively.

Equivalence

Let H is a $v \times v$ complex Hadamard matrix, and let U_1, U_2 be $v \times v$ monomial matrices (the nonzero entries of U_i are roots of 1, one per row/column).

Then $U_1 H U_2$ is a complex Hadamard matrix, *equivalent* to H. Moreover, $U_1 H U_2$ has an automorphism group 2-transitive on rows, iff H does.

Theorem. *Let H be a $v \times v$ complex Hadamard matrix admitting a group G of automorphisms permuting the rows of H 2-transitively. Then one of the following holds:*

(i)* $v = p^n$, H *is the character table of an elementary abelian group of order p^n. (Sylvester Hadamard for $p = 2$)*

(ii) $v = q + 1$, q *an odd prime power.* H *is of* Paley type. *(* iff $q \equiv 3 \mod 4$)*

(iii)* $v = 6$, $H = H_6$.

(iv)* $v = 36$, *example of Ito and Leon (1983) admitting $Sp(6,2)$.*

(v)* $v = 28$, *new(?) example admitting $\Gamma L(2,8)$. Corresponds to a 7-fold cover of $K_{28,28}$.*

(vi) $v = q^{2d}$, q *even.* G/N *is a known transitive subgroup of $Sp(2d,q)$,* N *regular.*

H *new(?) with entries $\pm 1, \pm \zeta$.*

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*Corresponds to a distance-regular cover.

New Example

(v) H_{28} is a 28×28 complex Hadamard matrix with entries in $\langle e^{2\pi i/7} \rangle$.

$H_{28} \longleftrightarrow \Gamma$, a distance-regular 7-fold cover of $K_{28,28}$.

H_{28} admits $G \cong \Sigma L(2,8) = SL(2,8){:}3$ permuting the 28 rows of H 2-transitively; G has orbit sizes 1,27 on columns.

G preserves a conic in $PG(2, \mathbb{F}_8)$, permuting the 28 passants ("exterior lines") 2-transitively. This yields a construction of H_{28}.

Previous results for an ordinary Hadamard matrix (entries ± 1) admitting a 2-transitive group G

W.M. Kantor (1969)—If the columns of H are not permuted faithfully by G, then H is a Sylvester Hadamard matrix.

N. Ito (1980)—Determination of H when G is almost simple.

Key Observation

Recall: an automorphism of H is a pair $g = (M_1, M_2)$ such that

$$M_1 H M_2^* = H.$$

Every group G of automorphisms of H has two monomial representations $\pi_1, \pi_2 : G \to GL(v, \mathbb{C})$ given by the projections

$$g = (M_1, M_2) \begin{array}{l} \overset{\pi_1}{\nearrow} \; M_1 \\ \\ \underset{\pi_2}{\searrow} \; M_2 \end{array}$$

Since H is invertible ($HH^* = vI$), π_1 and π_2 are equivalent.

If G permutes the rows of H 2-transitively, then π_1 (and hence also π_2) has at most two irreducible constituents. In particular, G has at most two orbits on columns of H.

Complex Hadamard Matrices arising "in nature"

Construction 1: Let

- G be group with two inequivalent permutation representations of degree v;
- L_1, L_2 the corresponding point stabilizers;
- $\theta_i : L_i \to \mathbb{C}^\times$ linear characters ($i = 1, 2$);
- $\theta_i^G : G \to GL(v, \mathbb{C})$ the induced (monomial) representations.

Then each θ_i^G has at most 2 irreducible constituents.

If θ_1^G and θ_2^G are equivalent irreducible representations, then they are intertwined by a complex Hadamard matrix H:

$$\theta_1^G(g)H = H\theta_2^G(g) \quad \forall g \in G.$$

G permutes the rows of H 2-transitively.

Construction 2: Let

- G be a 2-transitive permutation group of degree v;
- L the corresponding point stabilizer;
- $\theta : L \to \mathbb{C}^\times$ a linear character;
- $\theta^G : G \to GL(v, \mathbb{C})$ the induced (monomial) representation.

Suppose θ^G is reducible. Denote the constituent degrees by $v_1 + v_2 = v$. Then the centralizer of

$$\{\theta^G(g) : g \in G\}$$

is 2-dimensional, and contains a complex Hadamard matrix H iff

$$\frac{(v-1)(v_2 - v_1)^2}{v_1 v_2} \in \{0, 1, 2, 3, 4\}.$$

Details of Construction 2:

The centralizer of

$$\{\theta^G(g) : g \in G\}$$

is $\langle I, C\rangle_{\mathbb{C}}$ where

$C^* = C$;

entries of C are roots of 1, except 0's on main diagonal;

$C^2 = (v-1)I + \alpha C,\ \ \alpha^2 = \frac{(v-1)(v_2-v_1)^2}{v_1 v_2}$.

If $\alpha^2 \in \{0, 1, 2, 3, 4\}$ then $\alpha = \beta + \overline{\beta}$ for some root of unity β; choose

$$H = I - \beta C.$$

There are 2 or 3 more such constructions (similar).

We show that every example (G 2-transitive on the rows of H) must arise from one of these constructions.

We check which 2-transitive groups (obtained from the classification) are feasible.

Last Case (vi) (most technical)

- $v = 2^{2d} \geq 16$.
- $G = 2^{1+2d}{:}Sp(2d, 2)$, non-split over $Z(G) = \langle(-I, -I)\rangle$ of order 2. (Or replace $Sp(2d, 2)$ with an 'appropriate' subgroup.)

Choose subgroups $L_1, L_2 \cong 2 \times Sp(2d, 2)$ such that

- L_1 and L_2 are not conjugate in G;
- θ_1^G and θ_2^G are equivalent, where θ_i is the nonprincipal linear character of L_i.

Then

$$\{A : \theta_1^G(g)A = A\theta_2^G(g) \ \ \forall g \in G\} = \langle A_1, A_2\rangle_{\mathbb{C}}$$

contains complex Hadamard matrices

$$H_\zeta = A_1 + \zeta A_2, \quad \zeta \in \mathbb{C}^\times \text{ a root of 1.}$$