Cut Distance and Graphons

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(Ordinary) Graphs

Graphs are finite unless otherwise indicated.

Graphs are unlabelled unless otherwise indicated (so isomorphic graphs are considered the same).

In this talk, we consider only undirected graphs without loops or multiple edges

(although most of the theory generalizes beyond this special case).

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Example: Erdős-Rényi Random Graphs

Fix $0 < p < 1$. Choose a sequence of random graphs $G_n(p)$ with number of vertices $v_n \to \infty$ and each of the $\binom{v_n}{2}$ pairs of vertices joined with probability p independently at random.

Observations of $G_n(\frac{1}{2})$ with 10, 50 and 100 vertices:

The limit of such a sequence is the *graphon* $G_\infty(p)$ depicted by a unit square shaded with grayscale darkness p (the constant function $[0,1]^2 \to \{p\}$). We will make precise the topology in which the convergence $G_n(p) \to G_\infty(p)$ holds with probability 1.

The (deterministic) Paley graphs have the same limit: $P_q \to G_\infty(\frac{1}{2})$.

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What we *don't* want:

The graph with vertex set $\{0, 1, 2, 3, \ldots\}$, and pairs of vertices joined independently at random with uniform probability $p \in (0, 1)$, gives the Erdős-Rényi graph (studied first by Ackermann, and later Rado).

It is independent of the choice of $p \in (0, 1)$ (i.e. different choices of p give isomorphic graphs with probability 1).

Our precise notion of graph limit requires our use of a uniform probability measure on the vertex set.

For a finite vertex set, we use normalized counting measure. For a continuum of vertices $|V| = 2^{\aleph_0}$, typically $V = [0, 1]$, we use normalized Lebesgue measure. (There is no uniform probability measure on a countably infinite vertex set.)

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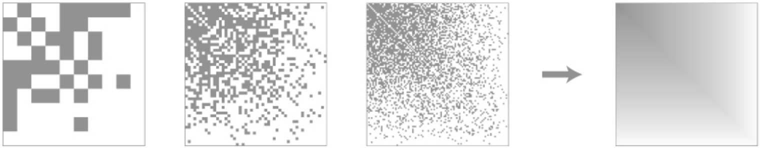
Example: Growing Uniform Attachment Graphs

Construct a sequence of graphs G_n (with vertex set $[n] = \{1, 2, \dots, n\}$) as follows, starting with $G_1 = \{\bullet\}$. For each $n \geqslant 2$, add a new vertex; and each pair $\{i, j\} \notin E(G_{n-1})$ is joined with probability $\frac{1}{n}$ (independently).

Observations of G_n for $n = 10, 50, 100, \infty$:



Here $G_\infty(x, y) = 1 - \max(x, y)$.

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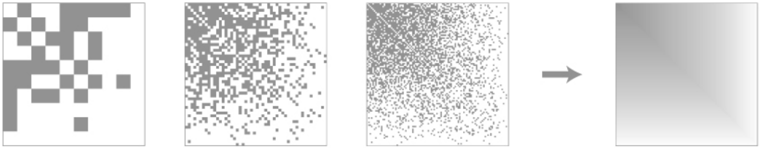

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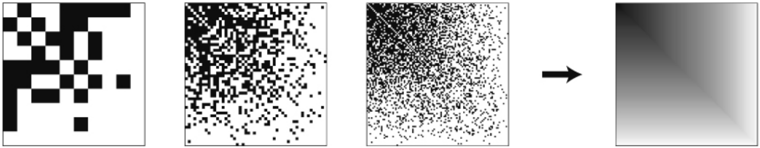

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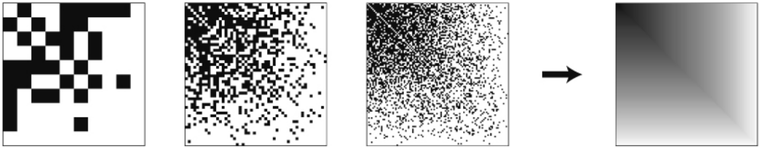

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Graph Homomorphisms

Let $G = (V, E)$ and $G' = (V', E')$ be graphs (vertex sets V, V'; edge sets $E \subseteq \binom{V}{2}$, $E' \subseteq \binom{V'}{2}$).

A homomorphism $G \to G'$ is a map $\phi : V \to V'$ which takes edges to edges (i.e. $\{x, y\} \in E \Rightarrow \{\phi(x), \phi(y)\} \in E'$).

The image $\phi(G) \subseteq G'$ is a subgraph (but not an induced subgraph in general).

If $\{x, y\} \in E \Leftrightarrow \{\phi(x), \phi(y)\} \in E'$, then the image is an induced subgraph.

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Homomorphism Density

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, and let $\hom(G, G')$ be the number of graph homomorphisms $G \to G'$.

Thus $\hom(\bullet, G') = |V'|$ and $\hom(\bullet\!\!-\!\!\bullet, G') = 2|E'|$.

The homomorphism density of G in G' is

$$t(G, G') = \text{probability that a random map } V \to V' \text{ is a homomorphism} = \frac{\hom(G, G')}{|V'|^{|V|}}.$$

If $|V'| = n$, then the edge density of G' is

$$2\frac{|E'|}{n(n-1)} = t(\bullet\!\!-\!\!\bullet, G') + O(\tfrac{1}{n}).$$

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Convergence of Graph Sequences

Let G_n be a sequence of finite graphs with number of vertices $v_n \to \infty$.

Suppose that for every finite graph F, the sequence of homomorphism densities $t(F, G_n)$ is Cauchy. Then $G_n \to W$ for the appropriate graphon $W = \lim_{n\to\infty} G_n$.

In place of the homomorphism density $t(F, G_n)$, one may use the induced subgraph density

$$t_{ind}(F, G_n) = \frac{\text{no. of embeddings } F \to G_n \text{ as an } \textit{induced subgraph}}{|V(G_n)|^{|V(F)|}}.$$

The space of graphons $\mathcal{G}$ modulo 'weak isomorphism', is the completion of the set of finite graphs. Here the appropriate distance is cut distance $\delta_\square$.

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Graphons

A labelled graphon may be viewed as a symmetric Lebesgue measurable function $W : [0,1]^2 \to [0,1]$. The symmetry requirement is that $W(x,y) = W(y,x)$. Identify two such functions if they agree almost everywhere.

If $\varphi : [0,1] \to [0,1]$ is a measure-preserving bijection, then we may relabel W as the graphon

$$W^{\varphi}(x,y) = W(\varphi(x), \varphi(y)).$$

A graphon is an equivalence class of graphons up to relabelling.

Given a graphon W, one obtains sequences of finite graphs $G_n \to W$ by sampling. Choose a sequence of positive integers $v_n \to \infty$; and let G_n be a graph with vertex set $[v_n] := \{1, 2, \ldots, v_n\}$. Choose $x_1, \ldots, x_{v_n} \in [0,1]$ independently and uniformly. Vertices $i \neq j$ are joined with probability $W(x_i, x_j)$ (independently for different pairs (i,j).)

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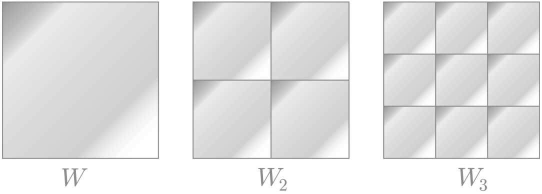
Define the cut distance between two graphons W and W' by

$$\delta_\Box(W,W') = \inf_{\varphi} \sup_{S,T\subseteq[0,1]} \int_{S\times T} \left| W'(x,y) - W(\varphi(x),\varphi(y)) \right| dx\,dy$$

where φ varies over all measure-preserving permutations of $[0,1]$ (in order that the distance is invariant under relabelling).

This is a *pseudometric*: There are pairs of graphons which are not isomorphic under relabelling, yet at distance zero; for example W, W_2, W_3 where $W_n(x,y) = W(nx - \lfloor nx \rfloor, ny - \lfloor ny \rfloor)$:



Distance Between Two Graphons

Define the cut distance between two graphons W and W' by

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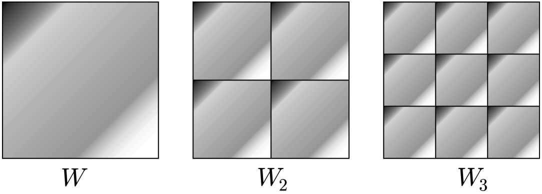

Distance Between Two Graphs on n vertices

Let G and G' be *labelled graphs* with the same vertex set $[n] = \{1, 2, \ldots, n\}$. Let $A = [a_{ij}]$ and $A' = [a'_{ij}]$ be their adjacency matrices.

The (normalized) edit distance between G and G' is

$$\frac{1}{n^2}\left|\{(i,j) \in [n]^2 \,:\, a_{ij} \neq a'_{ij}\}\right|.$$

The cut distance between the labelled graphs G and G' is

$$d_\square(G, G') = \frac{1}{n^2} \max_{S,T \subseteq [n]} \left|\{(i,j) \in S \times T \,:\, a_{ij} \neq a'_{ij}\}\right|.$$

For two arbitrary *unlabelled graphs* G and G', *both on n vertices*, we must minimize over all isomorphic graphs $\widehat{G} \cong G$ and $\widehat{G}' \cong G'$:

$$\widehat{\delta}_\square(G, G') = \min_{\widehat{G}, \widehat{G}'} d_\square(\widehat{G}, \widehat{G}').$$

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$$\tfrac{1}{n^2}\big|\{(i,j) \in [n]^2 \,:\, a_{ij} \neq a'_{ij}\}\big|.$$

The cut distance between the labelled graphs G and G' is

$$d_\square(G, G') = \tfrac{1}{n^2} \max_{S,T \subseteq [n]} \big|\{(i,j) \in S \times T \,:\, a_{ij} \neq a'_{ij}\}\big|.$$

For two arbitrary *unlabelled graphs* G and G', *both on n vertices*, we must minimize over all isomorphic graphs $\widehat{G} \cong G$ and $\widehat{G}' \cong G'$:

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Let G and G' be finite graphs on n and $n' = kn$ vertices respectively.

'Blow up' G by replacing 0's and 1's in its adjacency matrix, by $k \times k$ blocks of zeroes and ones respectively. The resulting graph $G(k)$ on $n' = nk$ vertices satisfies

$$t(F, G) = t(F, G(k))$$

for every finite graph F. Define the cut distance

$$\delta_\square(G, G') = \widehat{\delta}_\square(G(k), G').$$

Finally, if G and G' are arbitrary finite graphs on n and n' vertices respectively, choose k and k' such that $nk = n'k'$ and define

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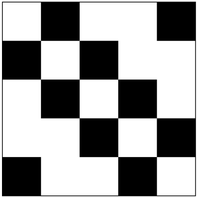
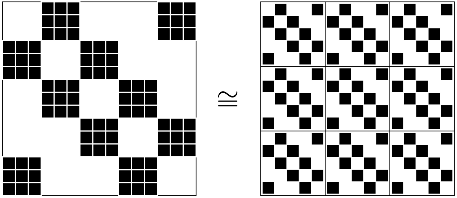
Cut Distance is a Pseudometric

For an arbitrary graph G and $k > 1$, the blow-up $G(k) \ncong G$, yet $\delta_\square(G(k), G) = 0$.

This should be compared with weak isomorphism of graphons. For example a 5-cycle G and its blow-up $G(3)$ are as shown:



G



$G(3)$

Consider a sequence of finite graphs converging to a graphon $W = \lim_{n\to\infty} G_n$. We view W as the kernel of an operator $T_W : L_2([0,1]) \to L_2([0,1])$:

$$(T_W f)(x) = \int_0^1 W(x,y) f(y)\, dy.$$

The operator T_W has (countable) real spectrum

$$-\lambda'_1 \leqslant -\lambda'_2 \leqslant -\lambda'_3 \leqslant \cdots \leqslant 0 \leqslant \cdots \leqslant \lambda_3 \leqslant \lambda_2 \leqslant \lambda_1$$

where $\lambda_k \to 0$ and $\lambda'_k \to 0$. Each G_n has (finite) real spectrum

$$-\lambda'_1(n) \leqslant -\lambda'_2(n) \leqslant -\lambda'_3(n) \leqslant \cdots \leqslant 0 \leqslant \cdots \leqslant \lambda_3(n) \leqslant \lambda_2(n) \leqslant \lambda_1(n).$$

Then $\lambda_k(n) \to \lambda_k$ and $\lambda'_k(n) \to \lambda'_k$ as $k \to \infty$.

Convergence of Spectra

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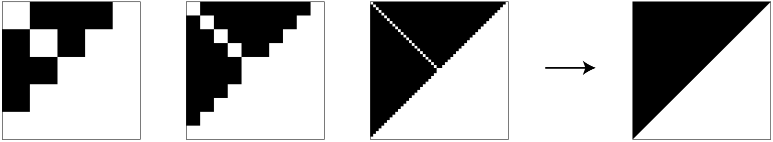
Let G_n have vertex set $[n]$. Join $i \in j$ iff $i + j \leqslant n$.

G_n for $n = 5, 10, 50, \infty$:



G_5 G_{10} G_{50} G_∞

The limit graphon $G_\infty(x, y) = \begin{cases} 1, & \text{if } x + y < 1; \\ 0, & \text{otherwise} \end{cases}$ is viewed as an actual *graph* with vertex set $[0, 1]$.

Example: Simple Threshold Graphs

Let G_n have vertex set $[n]$. Join $i \in j$ iff $i + j \leqslant n$.

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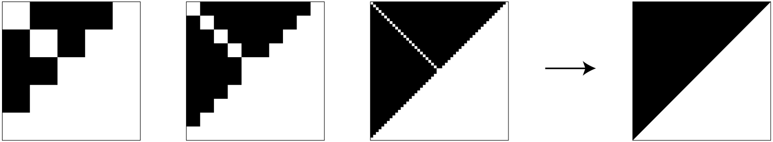


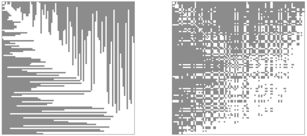
G_5 G_{10} G_{50} G_∞

The limit graphon $G_\infty(x, y) = \begin{cases} 1, & \text{if } x + y < 1; \\ 0, & \text{otherwise} \end{cases}$ is viewed as an actual *graph* with vertex set $[0, 1]$.

Example: Prefix Attachment Graphs

Let G_n have vertex set $[n]$. Start with $G_1 = \{\bullet\}$. For $n \geqslant 2$, join the new vertex n to every vertex in $[k] = \{1, 2, \ldots, k\}$ where $k \in [n-1]$ is chosen uniformly at random.

Observed G_{100} with vertices listed (a) in natural order, and (b) by decreasing degree:



$[0, 1]$ is an unnatural domain for the limit graphon! Better to use $V = [0, 1]^2$, and $W : V \times V \to [0, 1]$ is

$$W(x, y) = \begin{cases} 1, & \text{if } x_1 < x_2 y_2 \text{ or } x_2 < x_1 y_1; \\ 0, & \text{otherwise.} \end{cases}$$

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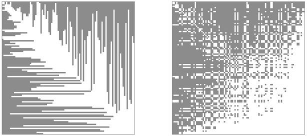

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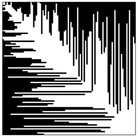 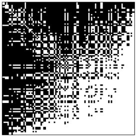

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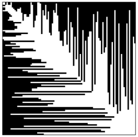

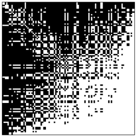

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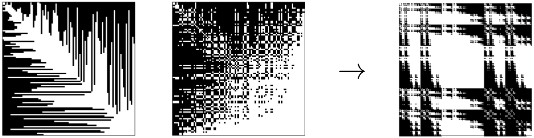

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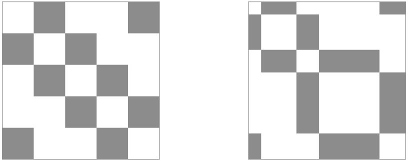
Example: Cloning

Start with an arbitrary graph G_1. For $n \geqslant 2$, form G_n from G_{n-1} by 'cloning' a randomly chosen vertex. (One new vertex is added; its neighbours are the same as those of the randomly chosen vertex of G_{n-1}.)

With probability 1, G_n converges to some graphon G_∞. But G_∞ is not uniquely determined! (Compare: the Pólya urn model.)

e.g. with a 5-cycle for G_1, one possible limit graph G_∞ is shown:



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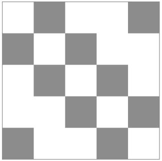

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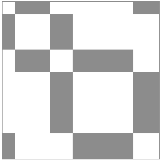

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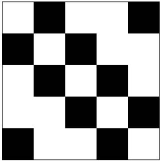

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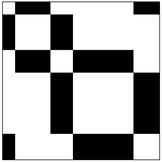

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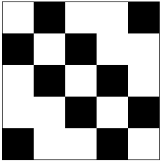

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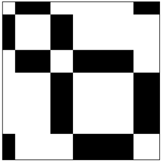

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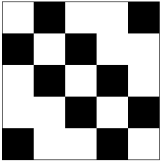

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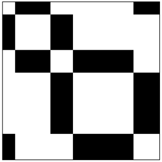

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Example: Growing Preferential Attachment Graphs

G_n has vertex set $[n]$. Start with $G_1 = \{\bullet\}$. For $n \geqslant 2$, form G_n from G_{n-1} by joining the new vertex n to $k \in [n{-}1]$ with probability $\frac{d_k+1}{n+1}$ (where d_k is the degree of vertex k in G_{n-1}).

With probability 1, G_n converges to a uniform graphon $G_\infty(x, y) = p$ for some $p \in [0, 1]$.

But the distribution of p is not known. (Every subinterval of $[0, 1]$ has positive probability.)

The graphs G_n have lower entropy (more 'clustering') than the Erdős-Rényi graphs $G_n(p)$.

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With probability 1, G_n converges to a uniform graphon $G_\infty(x, y) = p$ for some $p \in [0, 1]$.

But the distribution of p is not known. (Every subinterval of $[0, 1]$ has positive probability.)

The graphs G_n have lower entropy (more 'clustering') than the Erdős-Rényi graphs $G_n(p)$.

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