Embedding Finite Partial Linear Spaces in Finite Projective Planes

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A *partial linear space* (PLS) is a pair $\Gamma = (\mathfrak{P}, \mathfrak{L})$ consisting of a set $\mathfrak{P}$ (*points*) and a collection $\mathfrak{L}$ of distinguished subsets of $\mathfrak{P}$ (called *lines*) such that

(i) each line contains at least two points, and

(ii) any two distinct lines meet in at most one point.

A point-line pair (P, ℓ) in Γ is called a *flag* or an *antiflag* according as $P \in \ell$ or $P \notin \ell$.

Let $\Gamma = (\mathfrak{P}, \mathfrak{L})$ and $\tilde{\Gamma} = (\tilde{\mathfrak{P}}, \tilde{\mathfrak{L}})$ be two partial linear spaces.

An *embedding* $\alpha : \Gamma \to \tilde{\Gamma}$ is a pair of injections

$$\alpha_1 : \mathfrak{P} \to \tilde{\mathfrak{P}}, \qquad \alpha_2 : \mathfrak{L} \to \tilde{\mathfrak{L}}$$

such that for all $P \in \mathfrak{P}$, $\ell \in \mathfrak{L}$,

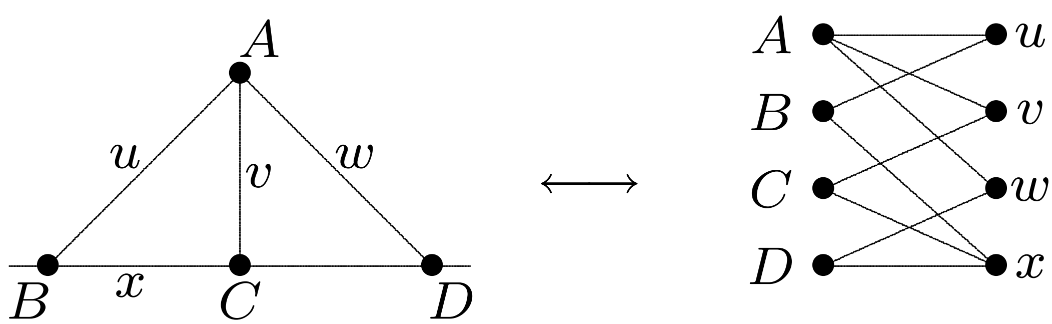
$$P \in \ell \;\Rightarrow\; \alpha_1(P) \in \alpha_2(\ell).$$

Such an embedding is *strong* if

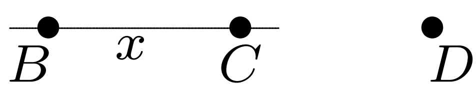
$$P \in \ell \iff \alpha_1(P) \in \alpha_2(\ell).$$

Every incidence system may be identified with its *point-line incidence graph*.

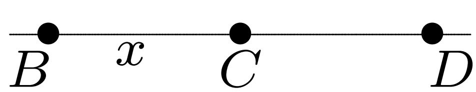
Thus a PLS corresponds to a bipartite graph of girth ≥ 6 (i.e. no 4-cycles).



An embedded PLS corresponds to a subgraph.



A strongly embedded PLS corresponds to an induced subgraph.



Theorem. *Every PLS is strongly embeddable in an infinite projective plane.*

Proof. Use free closure. □

Question. *Can every finite PLS be embedded in a finite projective plane?*

Proposition. *The following two statements are logically equivalent.*

(i) Every finite PLS is embeddable in a finite projective plane.

(ii) Every finite PLS is strongly embeddable in a finite projective plane.

Our Survey Says. . .

Francis Buekenhout polled participants at a recent conference if they believed that every finite linear space is embeddable in a finite projective plane.

22 voted YES

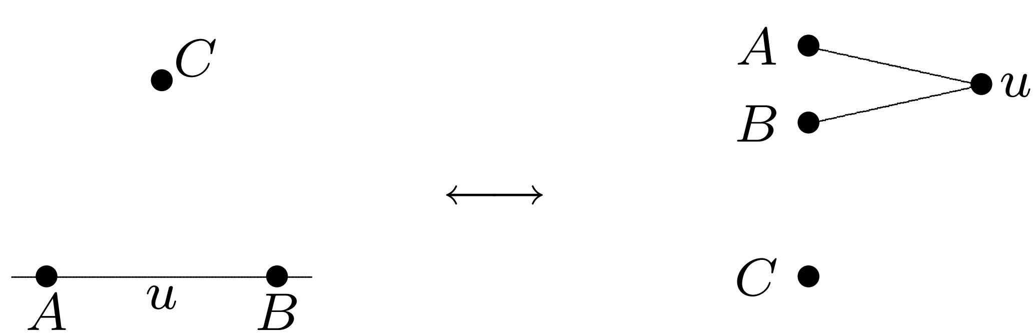
2 voted NO

19 abstained

A partial linear space Γ is *saturated* if no edges can be added to its incidence graph without creating a 4-cycle.

Lemma. *Every finite PLS Γ is strongly embeddable in a saturated finite PLS.*

Proof. Join every antiflag (P, ℓ) in Γ by a path of length three (using two new vertices per antiflag). Then add further edges (as necessary) until the graph is saturated. $\square$



Proposition. *The following two statements are logically equivalent.*

(i) Every finite PLS is embeddable in a finite projective plane.

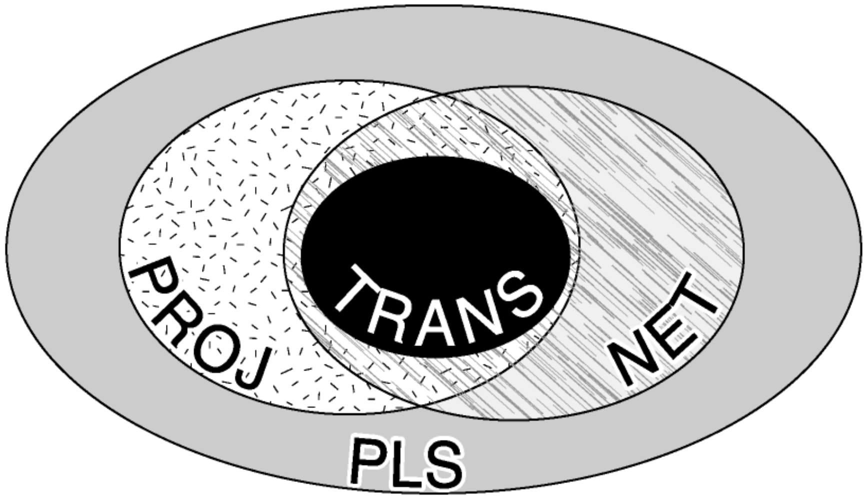
(ii) Every finite PLS is strongly embeddable in a finite projective plane.

Proof. (ii)⇒(i) is trivial.

(i)⇒(ii): Let Γ be a finite PLS. First strongly embed $\Gamma \to \Gamma_1$ where Γ_1 is a saturated finite PLS. Then embed $\Gamma_1 \to \Pi$ where Π is a finite projective plane. The composite

$$\Gamma \to \Gamma_1 \to \Pi$$

is necessarily a strong embedding of Γ. □



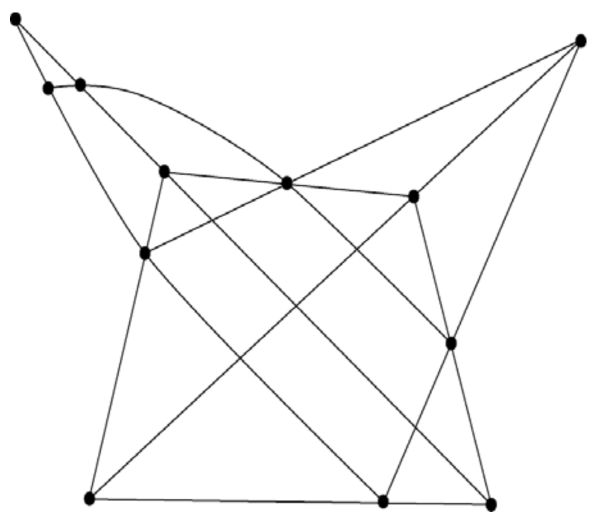
PLS: the class of all finite partial linear spaces;

PROJ: the class of all point-line incidence systems formed by subsets of finite projective planes;

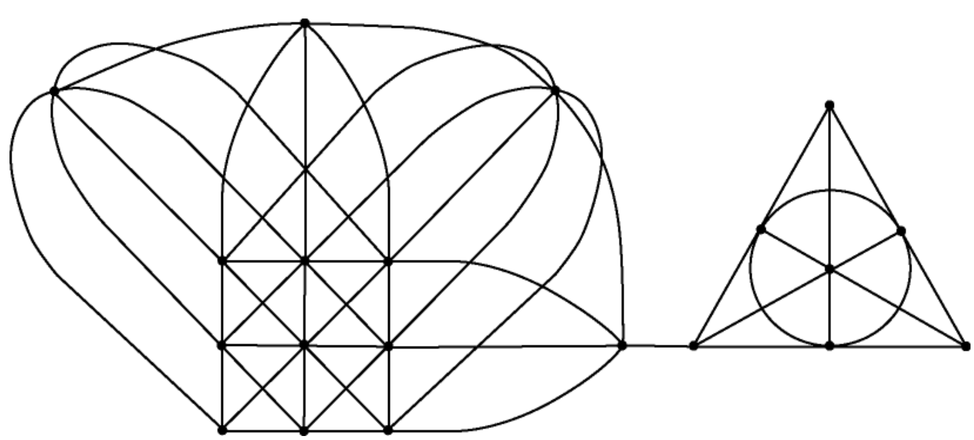
TRANS: the class of all point-line incidence systems formed by subsets of finite projective translation planes;

NET: the class of all point-line incidence systems formed by subsets of finite translation nets (arising from partial spreads).

A finite PLS which *cannot* be embedded in any *Desarguesian* plane:



Another:



Theorem. *For every $d \geq 1$, there exists a finite PLS Γ which is not embeddable in any André plane of dimension $\leq d$ over its kernel.*

André Planes

Let $E \supset F$ be an extension of finite fields, $|E| = q^d$, $|F| = q$.

The automorphism group

$$Aut(E/F) = \{1, \sigma, \sigma^2, \ldots, \sigma^{d-1}\}$$

where $\sigma : E \to E$, $x \mapsto x^q$.

The norm map $N : E \to F$, $x \mapsto x^{1+q+q^2+\cdots+q^{d-1}}$

Chose *any map* $\phi : F^\times \to Aut(E/F)$ s.t. $\phi(1)=1$. There are d^{q-2} choices for ϕ.

Each ϕ gives an *André plane* with

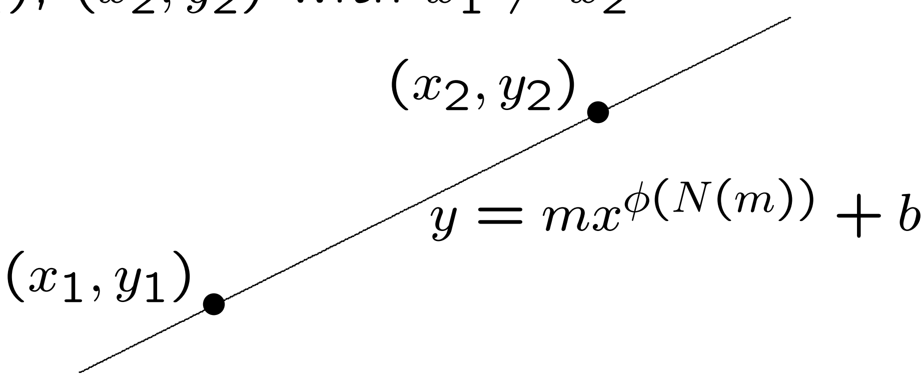
q^{2d} points $(x, y) \in E \times E$;

$q^d(q^d + 1)$ lines $x = a$ (for $a \in E$);

$$y = mx^{\phi(N(m))} + b \quad \text{(for } m, b \in E\text{)}.$$

Essence of Proof that this is a plane

Try to find a line through two given points (x_1, y_1), (x_2, y_2) with $x_1 \neq x_2$



$$y_2 - y_1 = m(x_2 - x_1)^{\phi(N(m))}$$

$$\Rightarrow N(y_2 - y_1) = N(m)N(x_2 - x_1)$$

$$\Rightarrow N(m) = \frac{N(y_2 - y_1)}{N(x_2 - x_1)} \text{ is determined}$$

$$\Rightarrow m = \frac{y_2 - y_1}{(x_2 - x_1)^{\phi(N(m))}} \text{ is determined}$$

$$\Rightarrow b \text{ is determined}$$

Theorem. *For every $d \geq 1$, there exists a finite PLS Γ which is not embeddable in any André plane of dimension $\leq d$ over its kernel.*

Lemma. *Given a finite PLS Γ and an integer $d \geq 1$, there exists a finite PLS $\widehat{\Gamma}$ such that for every d-colouring of the lines, there is an embedded copy of Γ with all lines having the same colour.*

Proof. This follows from a result of Nešetřil and Rödl. $\square$

Proof of Theorem. Let Γ be a finite PLS not embeddable in any Desarguesian plane. Let $\widehat{\Gamma}$ be as in the Lemma. Suppose $\widehat{\Gamma}$ embeds in an André plane $\mathcal{A}$ as above.

Then Γ is embedded in $\mathcal{A}$ in such a way that for some fixed $\alpha \in Aut(E/F)$, all lines of $\Gamma \subset \mathcal{A}$ have the form $y = mx^{\alpha} + b$ for some $m, b \in E$.

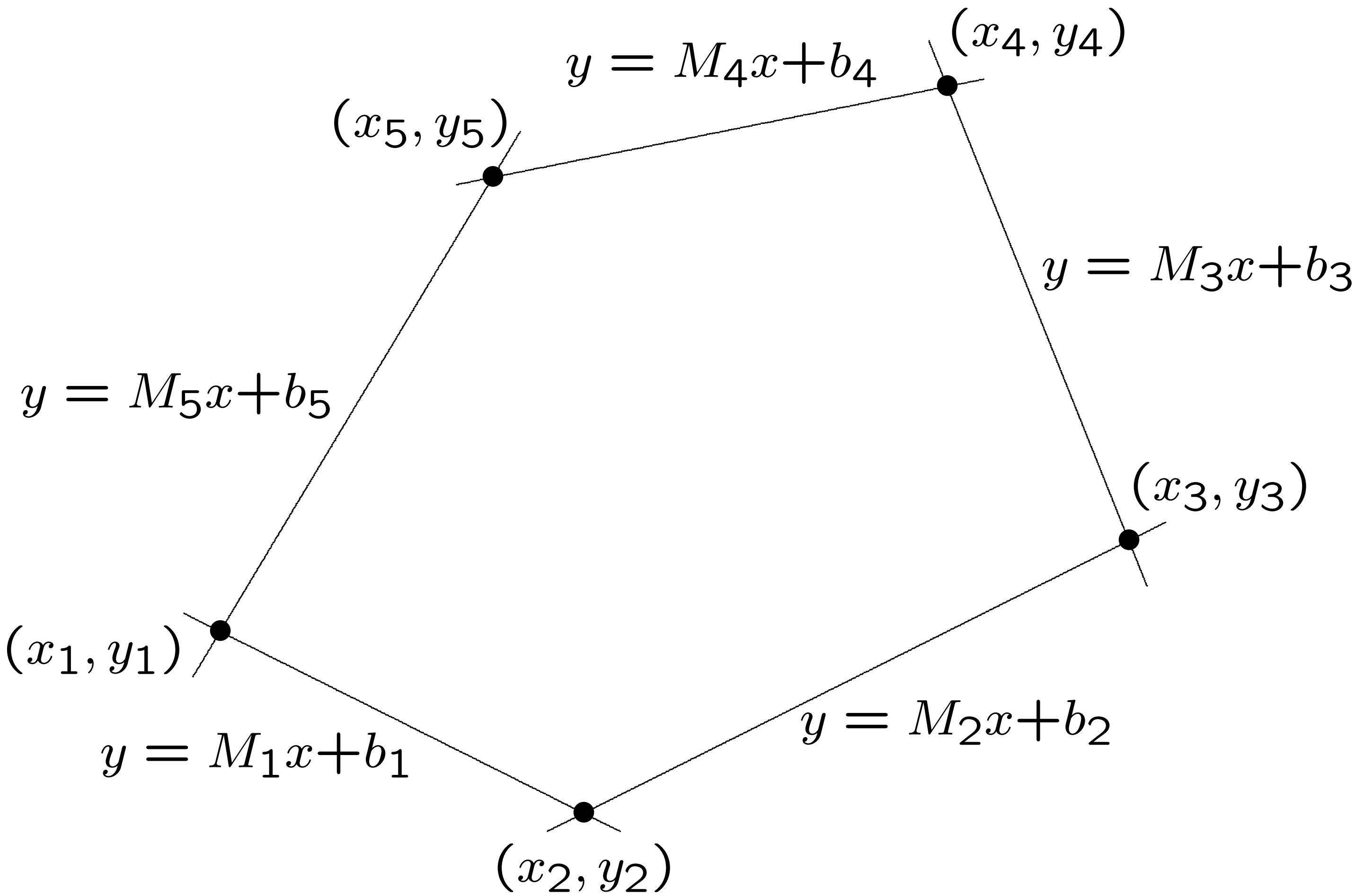
Apply $(x, y) \mapsto (x^{\alpha^{-1}}, y)$ to give an embedding of Γ in a Desarguesian net, a contradiction. $\square$

Open Questions

Does there exist a finite PLS which is not embeddable in any translation net of dimension 2 over its kernel?

Does there exist a finite PLS which is not embeddable in *any* André plane?

Is there a good criterion for embeddability of a finite PLS in a finite Desarguesian plane?



$$M_1x_1 - M_1x_2 + M_2x_2 - M_2x_3 + M_3x_3 \\ -M_3x_4+M_4x_4-M_4x_5+M_5x_5-M_5x_1 = 0$$

Let $\Gamma = (\mathfrak{P}, \mathfrak{L})$ be a finite PLS. A reasonable approach to embedding Γ in a finite translation net follows:

$C_0(\Gamma) =$ the $\mathbb{Z}$-module freely generated by $\mathfrak{P} \cup \mathfrak{L}$

$C_1(\Gamma) =$ the $\mathbb{Z}$-module freely generated by the flags of Γ

Consider the complex

$$0 \xrightarrow{\delta} C_1(\Gamma) \xrightarrow{\delta} C_0(\Gamma) \xrightarrow{\delta} 0$$

where $\delta(P, \ell) = P - \ell$ for each flag (P, ℓ). The Euler characteristic of this complex is

$$\dim H_0(\Gamma) - \dim H_1(\Gamma) = \dim C_0(\Gamma) - \dim C_1(\Gamma)$$

Here

$\dim H_0(\Gamma) =$ number of connected components;

$\dim H_1(\Gamma) =$ dimension of the circuit space;

$\dim C_0(\Gamma) =$ total number of points and lines;

$\dim C_1(\Gamma) =$ number of flags of Γ.

Given Γ, we want to find a finite vector space V such that Γ embeds in an affine translation net in $V \oplus V$. We seek functions

$$f : \mathfrak{P} \to V \quad \text{and} \quad g : \mathfrak{L} \to E := End(V)$$

such that

(i) f is injective;

(ii) for all $\ell \neq \ell'$ in $\mathfrak{L}$, we have $g(\ell) - g(\ell') \in E^\times = GL(V)$; and

(iii) $H_1(\Gamma) = Z_1(\Gamma)$ is contained in the kernel of the linear map

$$f \times g : C_1(\Gamma) \to V$$

defined by $(P, \ell) \mapsto g(\ell)f(P)$ for every flag (P, ℓ) of Γ.

This yields a translation net (i.e. partial spread) of $V \oplus V$ with components

$$S_\ell := \{(v, g(\ell)v) : v \in V\} \text{ for } \ell \in \mathfrak{L}$$

in which Γ is embedded.