Deflation in Coxeter Groups

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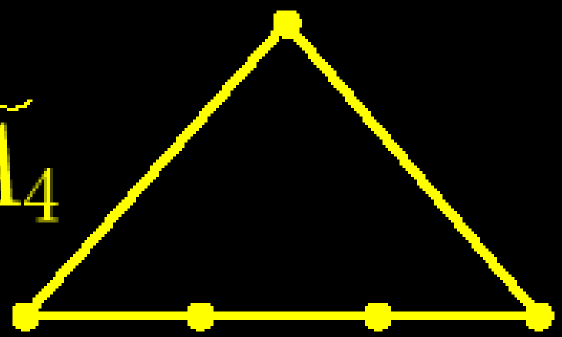
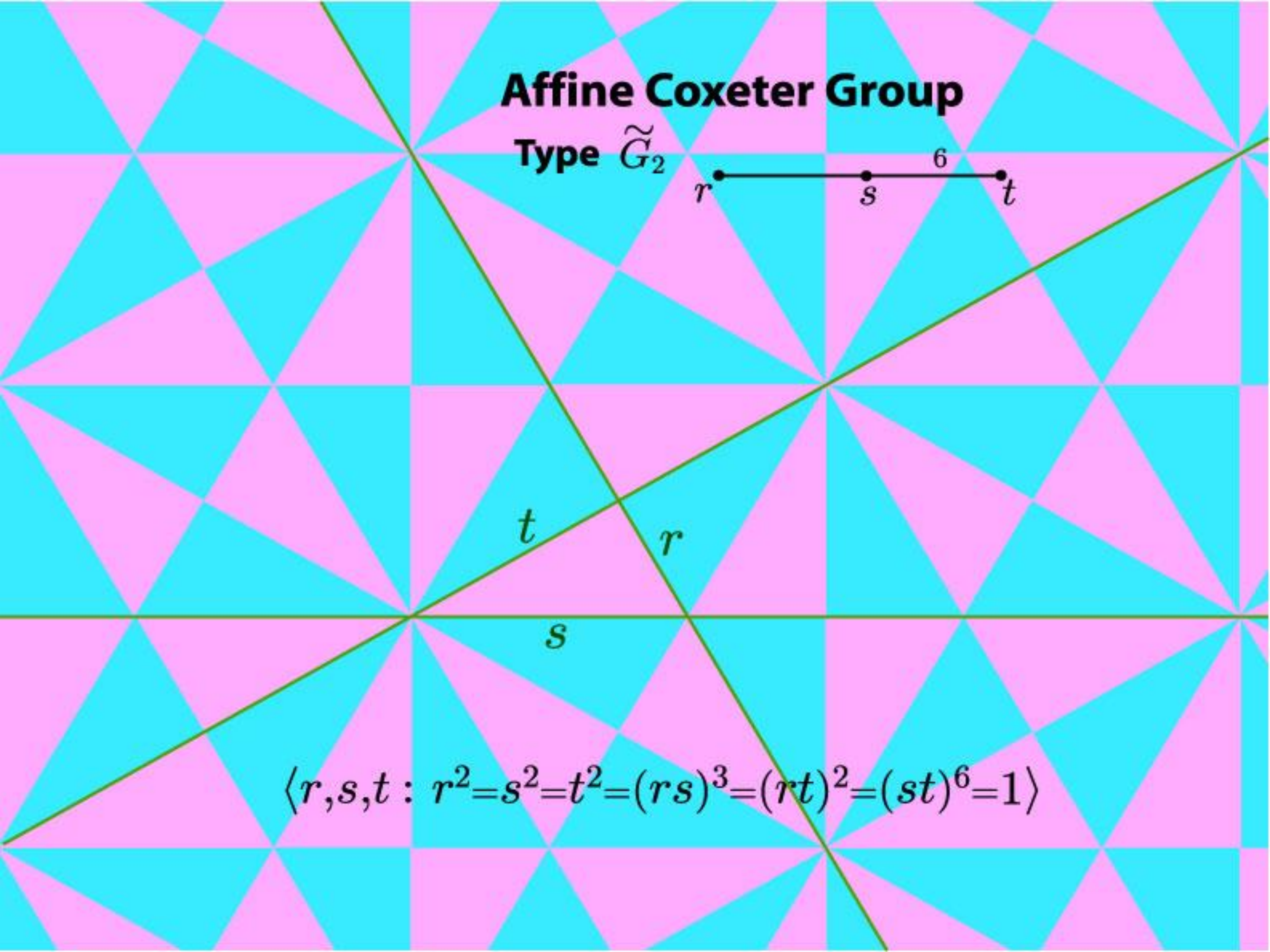
$\tilde{A}_4$



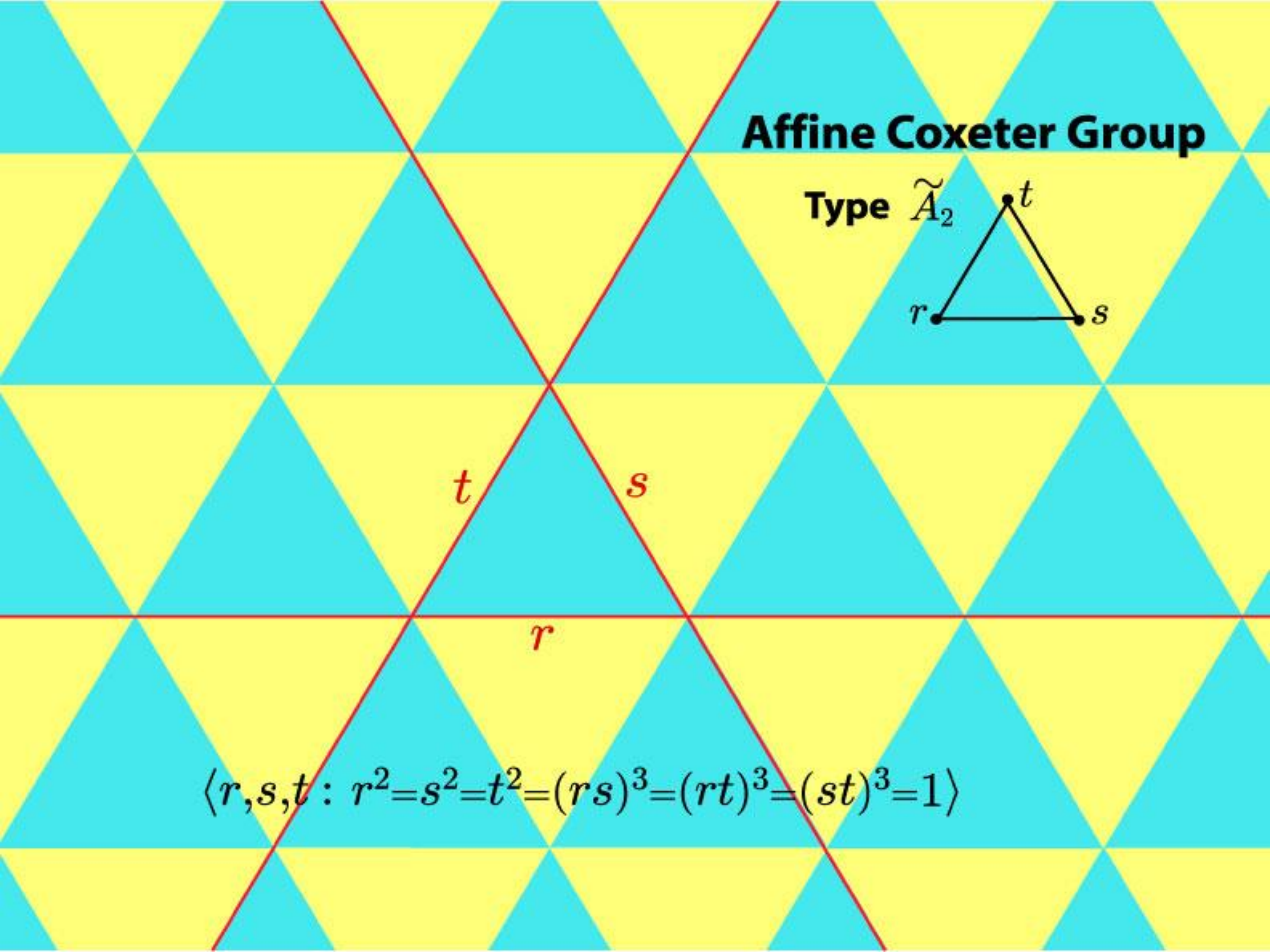
based on recent work (1993-present) of
John H. Conway and Christopher S. Simons

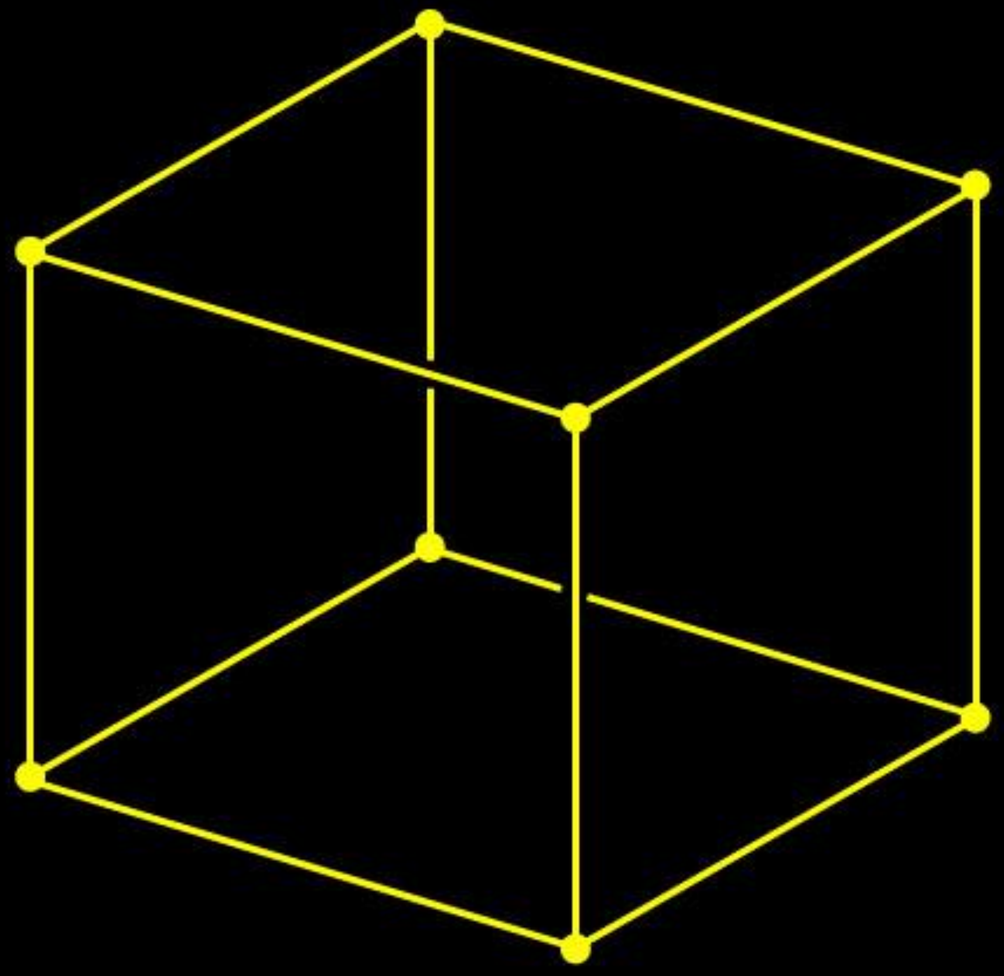
Affine Coxeter Group

Type $\widetilde{G}_2$

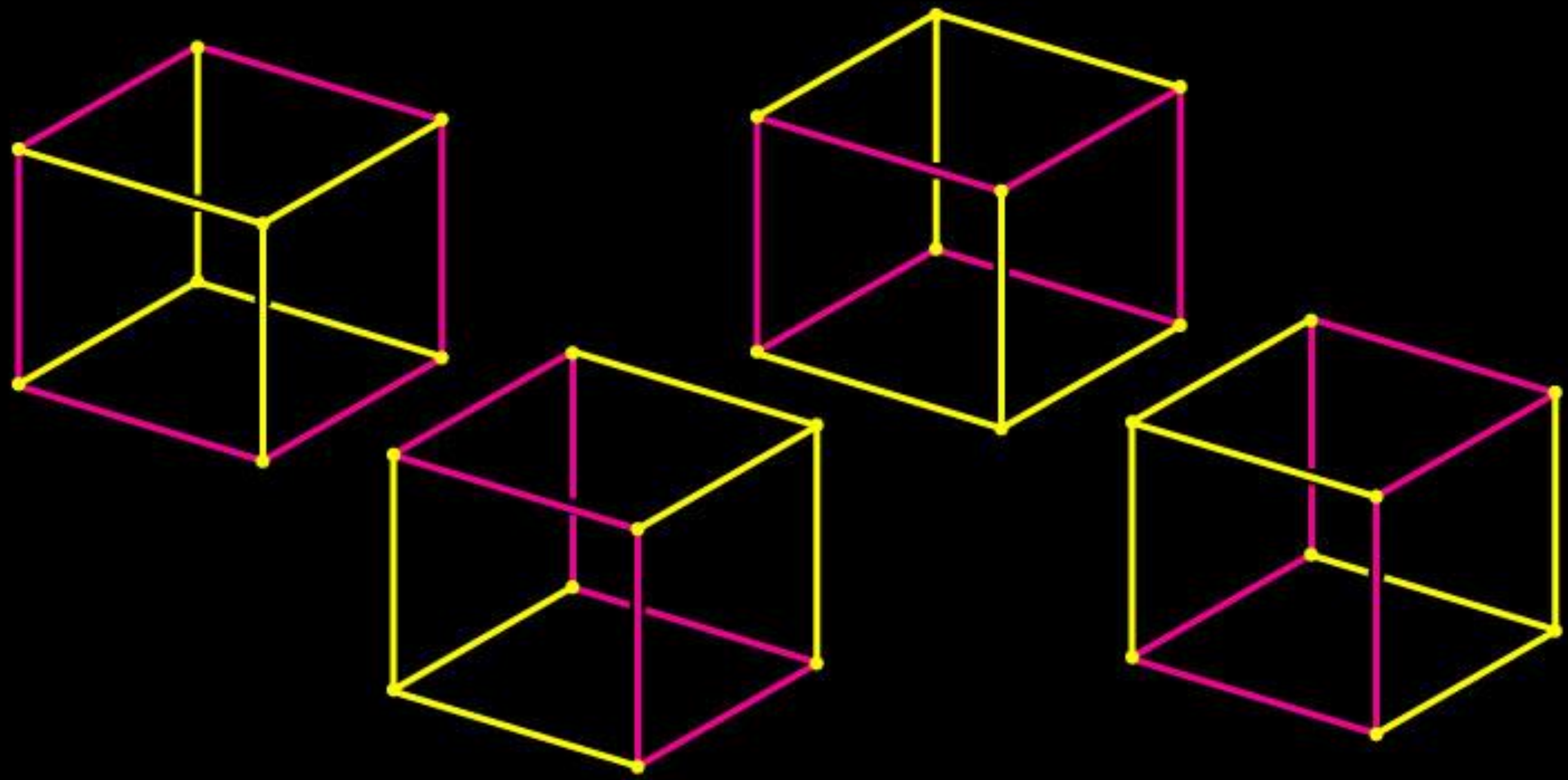


$$\langle r, s, t : r^2 = s^2 = t^2 = (rs)^3 = (rt)^2 = (st)^6 = 1 \rangle$$

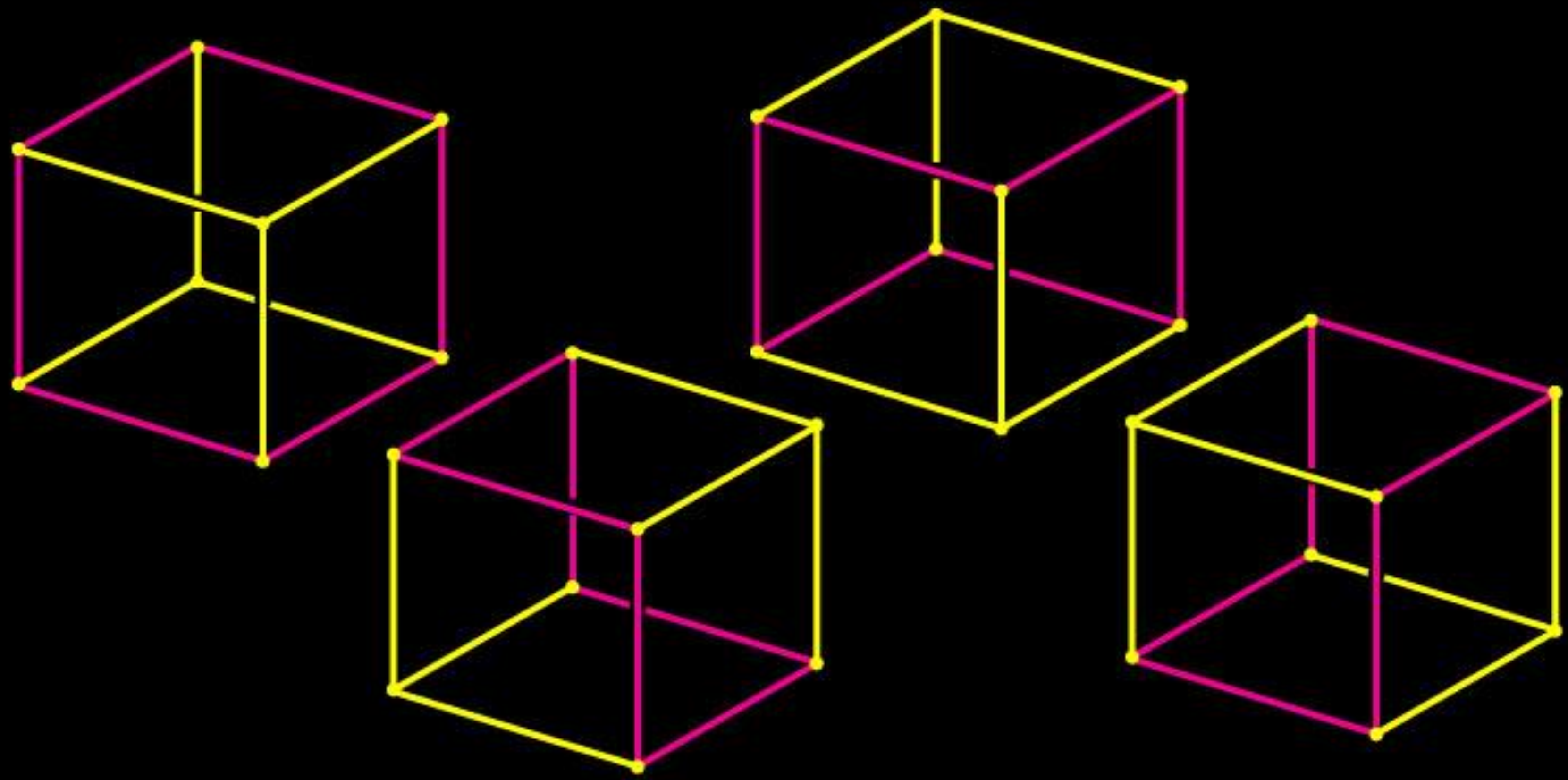

Affine Coxeter Group
Type $\widetilde{A}_2$
t
r
s
t
s
r
$\langle r,s,t : r^2 = s^2 = t^2 = (rs)^3 = (rt)^3 = (st)^3 = 1 \rangle$



The cube defines an infinite Coxeter group.



The cube has four 6-cycles (as induced subgraphs).

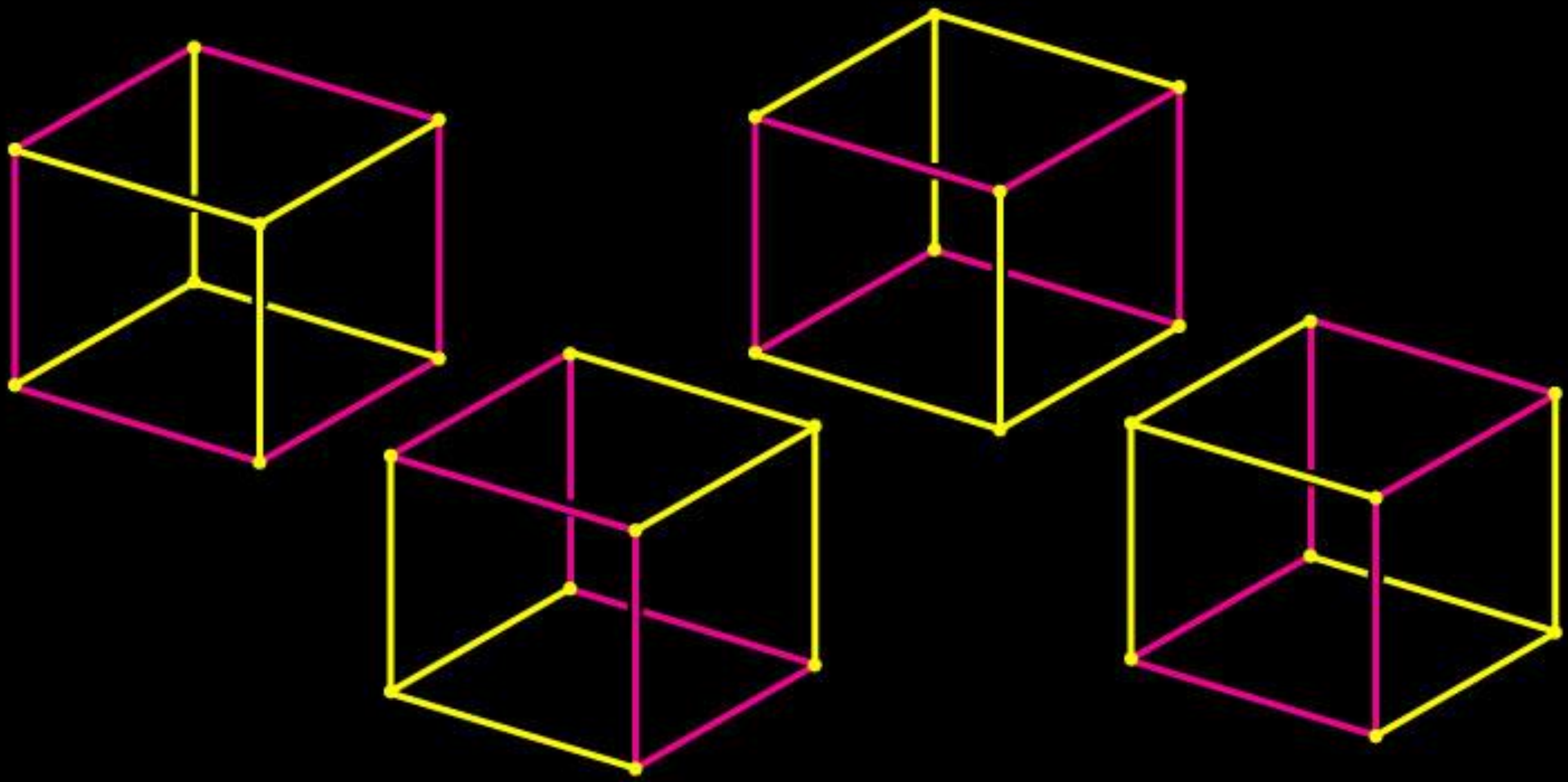


Deflate every $\widetilde{A}_5$ to an A_5 .

This can be done by imposing an additional relation

$$s_1 s_2 s_3 s_4 s_5 s_6 s_5 s_4 s_3 s_2 = 1$$

for every 6-cycle.



Deflate every $\widetilde{A}_5$ to an A_5 .

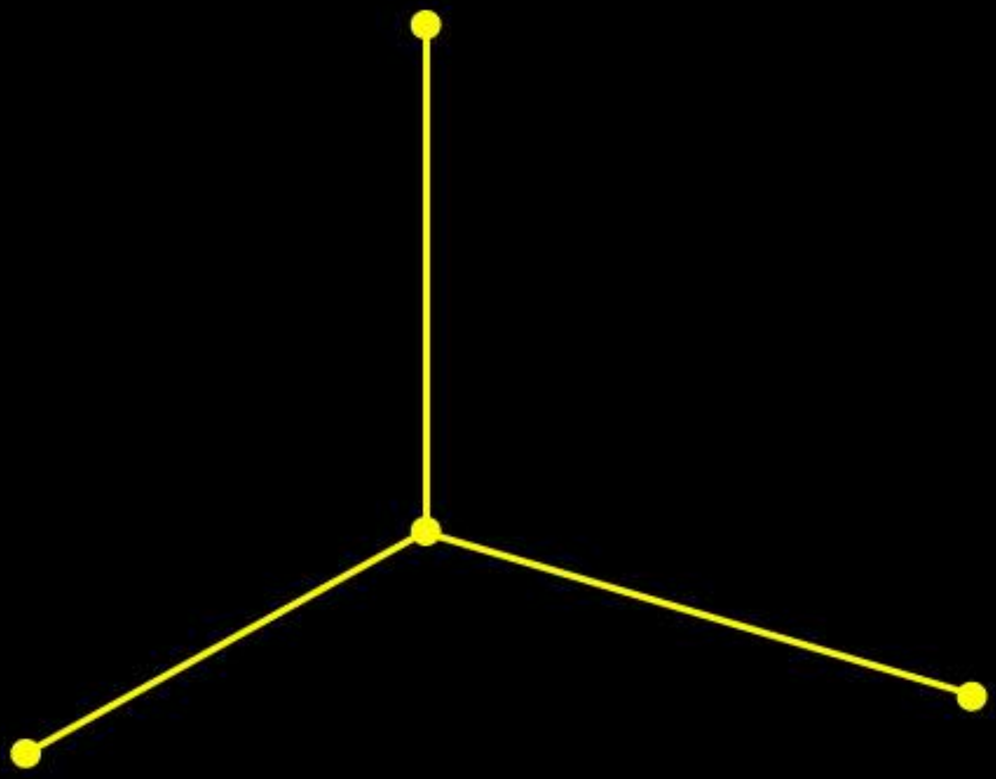
The resulting homomorphic image is a finite group isomorphic to $PSp_4(3)\times 2 = U_4(2)\times 2$ of order 51840.

Let G be a finite group generated by a set of n elements of order 2.

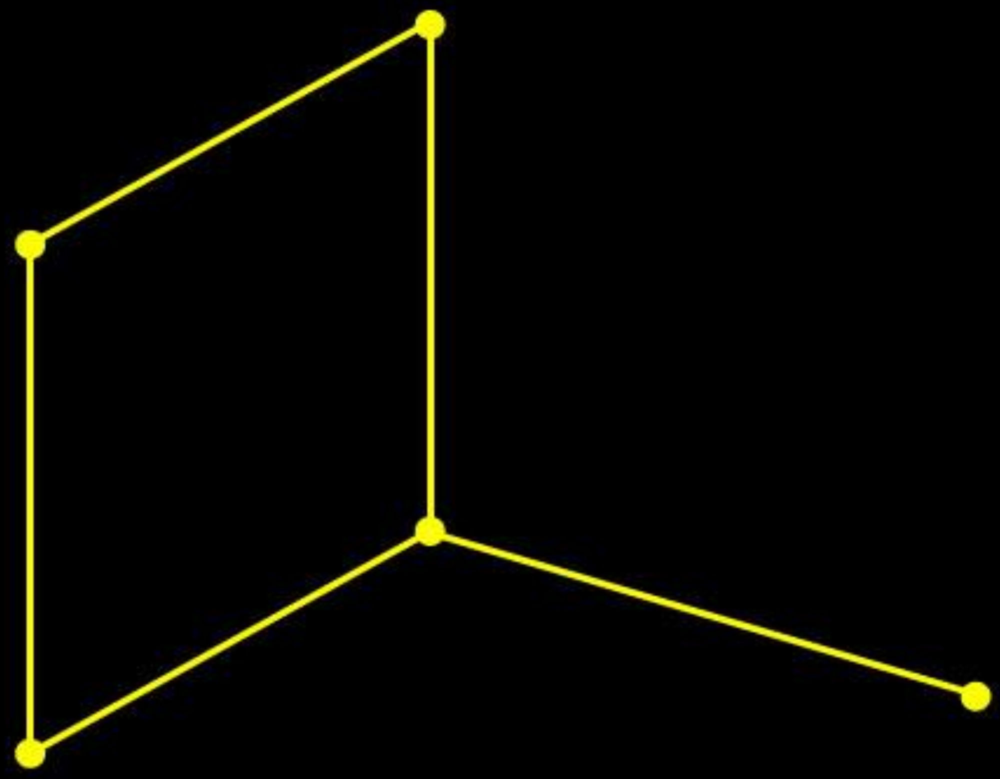
Then G is a homomorphic image of a Coxeter group defined by a graph on n vertices. But this presentation is not usually very concise. We would prefer n to be very small relative to $|G|$.

Problem: Find groups G having a concise presentation obtained by deflating a Coxeter group.

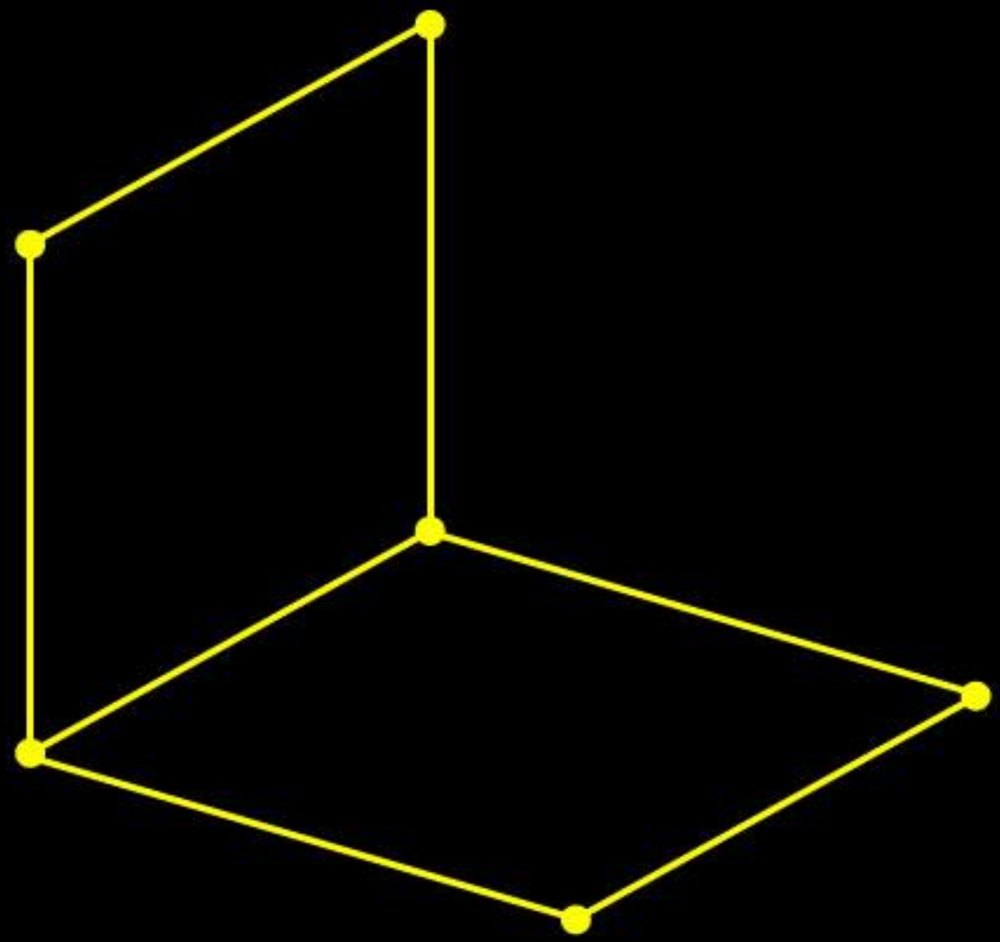
Examples grow on trees!

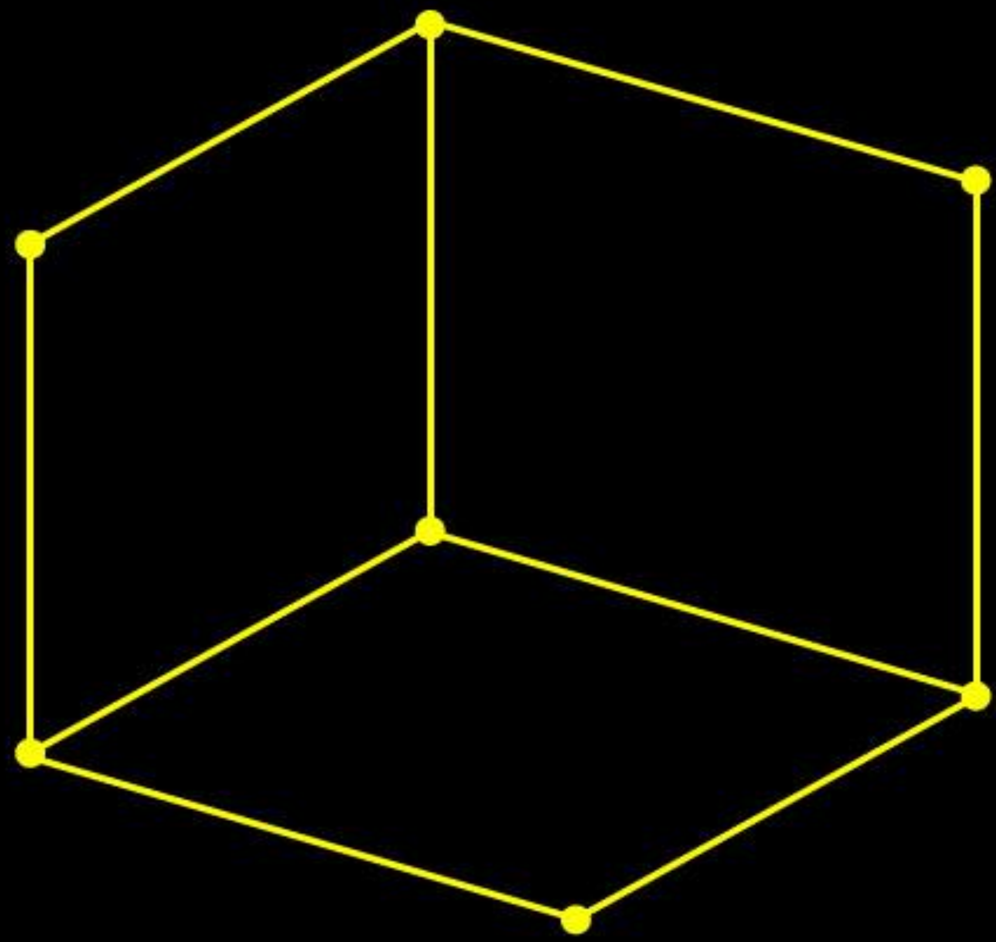


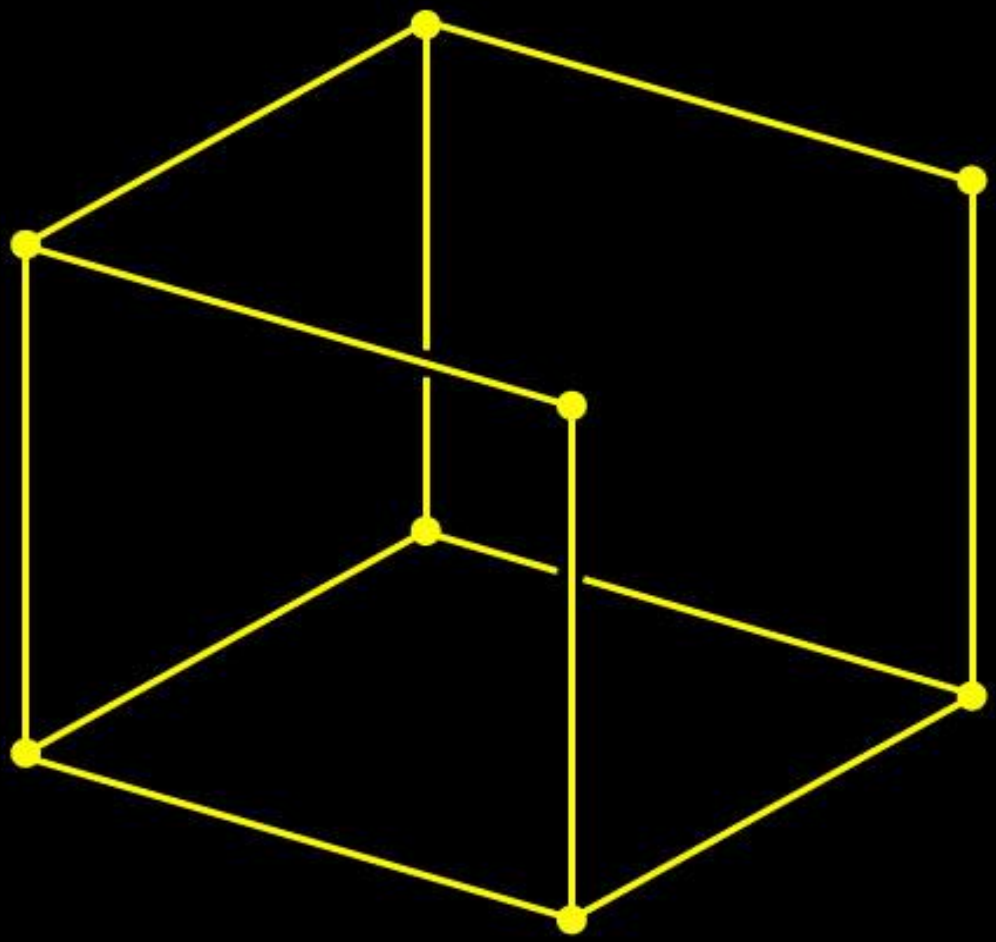
The graph Y_{111}

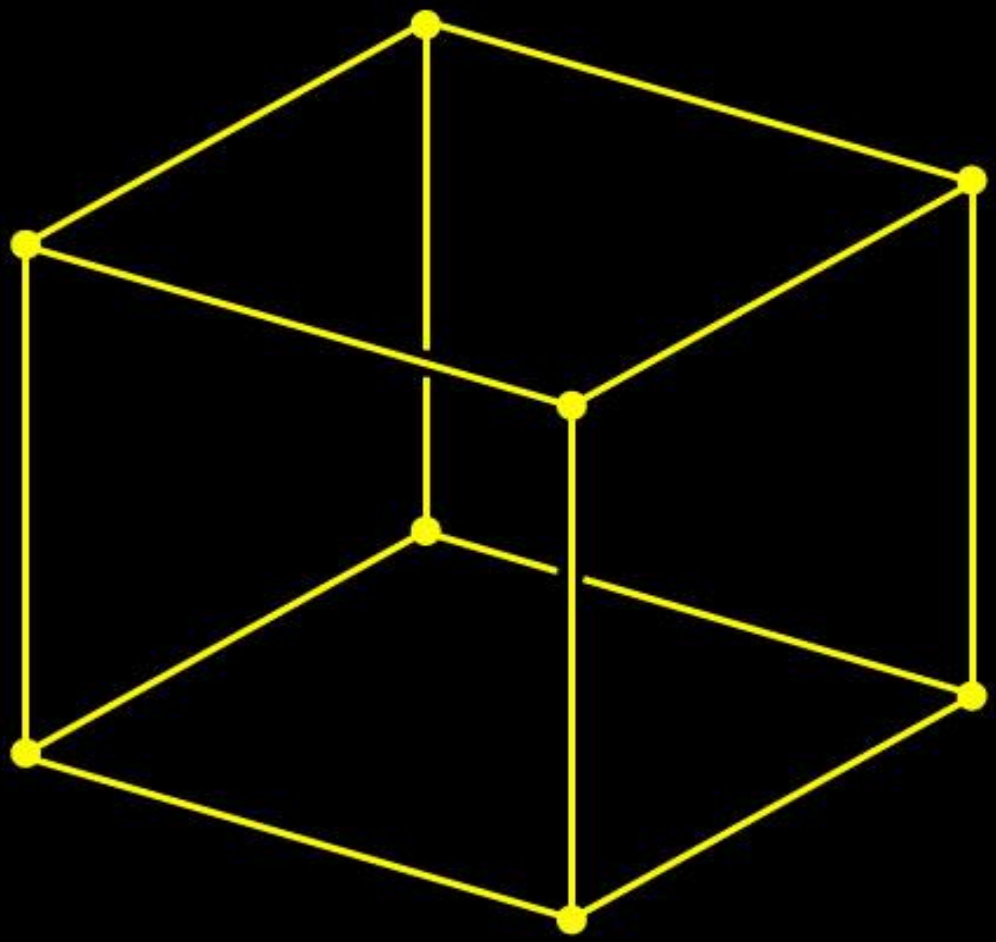


Close A_3 to form $\widetilde{A}_3$









Goal: Close every 3-path to a 4-cycle, *adding as few vertices as possible.*

Interesting examples arise also from:

Y_{222} (7 vertices)

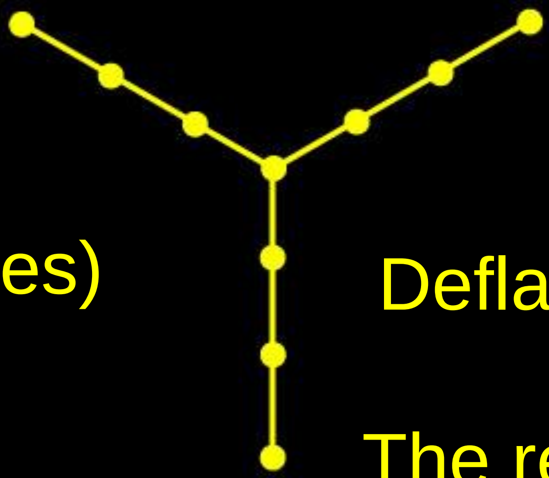
Close induced
5-paths to 6-cycles

Yields the Petersen
graph (10 vertices)

Deflate every $\tilde{A}_6$ to A_6

The resulting group is
$PSp_4(3){:}2 = U_4(2){:}2$
of order 51840

Interesting examples arise also from:



Y_{333} (10 vertices)

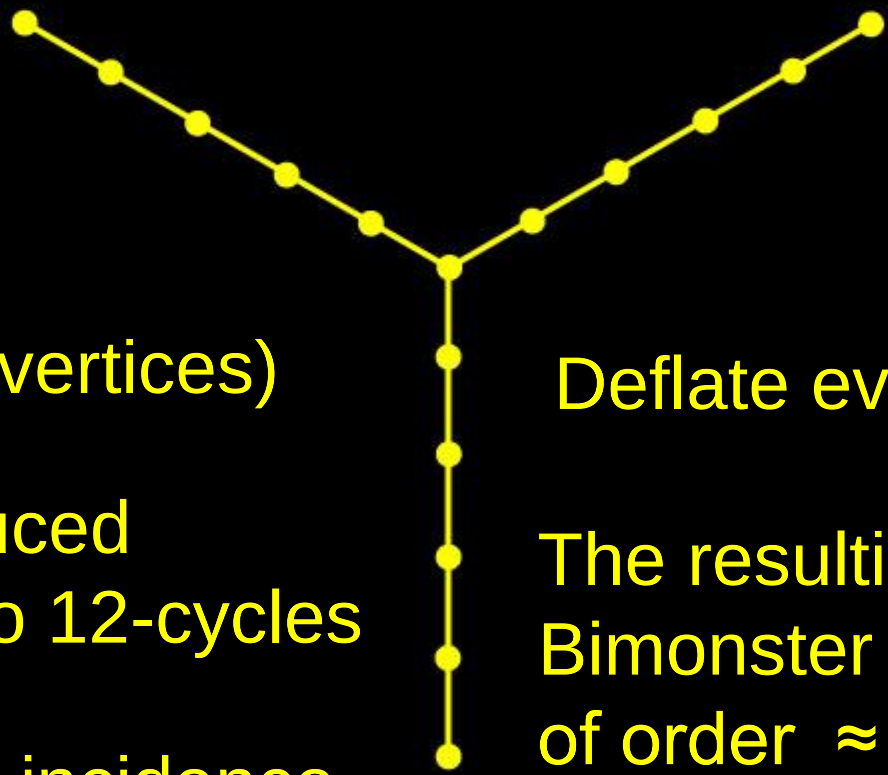
Close induced
7-paths to 8-cycles

Yields the incidence
graph of $PG_2(2)$
(14 vertices)

Deflate every $\widetilde{A}_8$ to A_8

The resulting group is the
group $O_8^-(2)$:2
of order 394,813,440

Interesting examples arise also from:



Y_{555} (16 vertices)

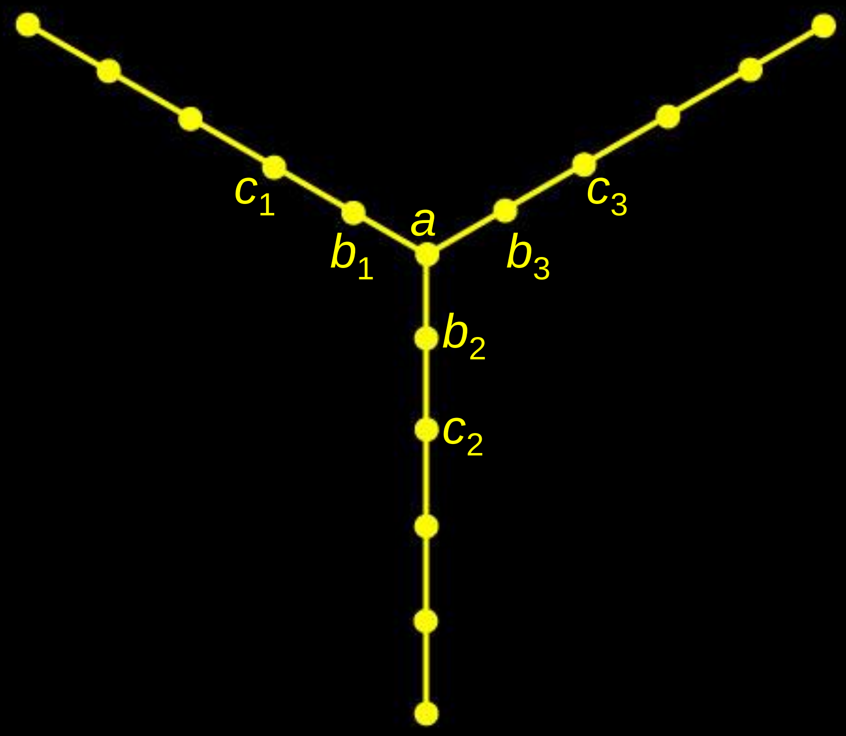
Close induced 11-paths to 12-cycles

Yields the incidence graph of $PG_2(3)$ (26 vertices)

Deflate every $\widetilde{A}_{12}$ to A_{12}

The resulting group is the Bimonster $(M \times M):2$ of order $\approx 1.31 \times 10^{108}$

Theorem (Ivanov-Norton, 1992) The Bimonster $(M \times M):2$ is presented by the Coxeter group of the graph Y_{555} with the single additional relation $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$.



$V = \mathbb{R}^{3k+1}$ is the set of all vectors

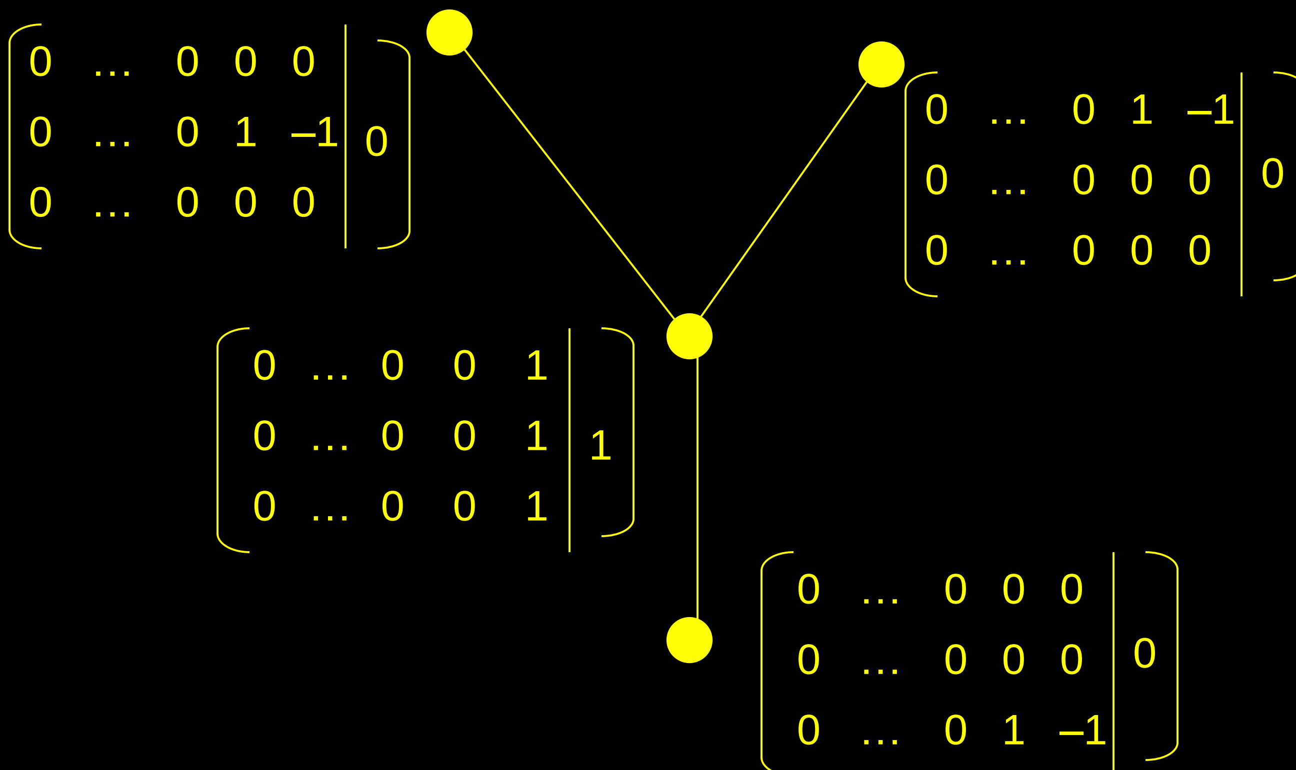
$$v = \left(\begin{array}{cccc|c} v_{11} & v_{12} & \cdots & v_{1k} & \\ v_{21} & v_{12} & \cdots & v_{2k} & v_\infty \\ v_{31} & v_{32} & \cdots & v_{3k} & \end{array} \right)$$

with the Lorentzian inner product

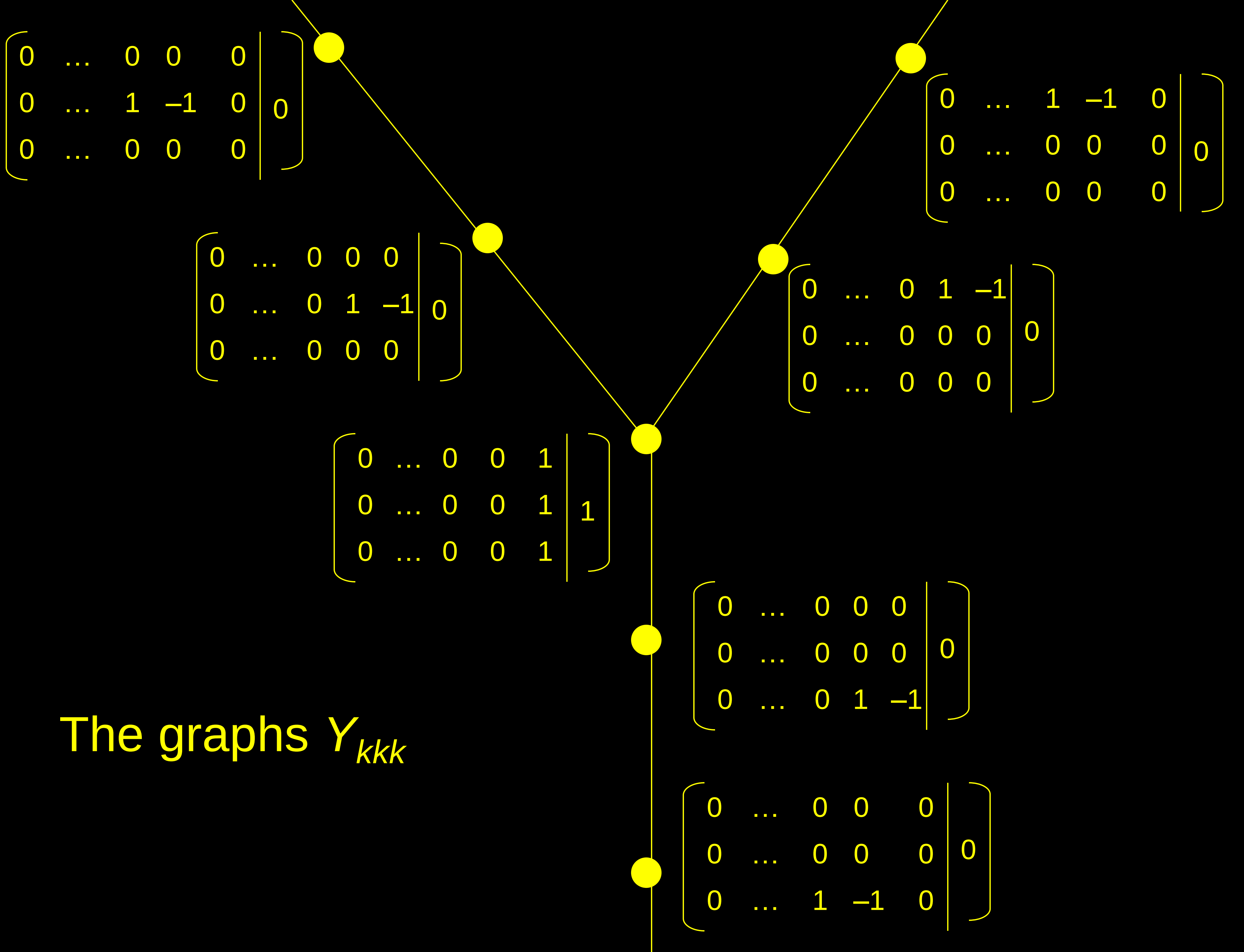
$$v \bullet w = v_{11}w_{11} + v_{12}w_{12} + \ldots + v_{3k}w_{3k} - v_\infty w_\infty$$

Vertices are represented by vectors $v_1, v_2, \ldots, v_n$ such that $v_i \bullet v_j = 2$. Vertices i,j are joined whenever $v_i \bullet v_j = \pm 1$ (but possibly for other nonzero values of $v_i \bullet v_j$ as well).

$$\left(\begin{array}{ccccc|c} 0 & \dots & 0 & 0 & 1 & \\ 0 & \dots & 0 & 0 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 & \end{array}\right)$$


0 … 0 0 0
0 … 0 1 –1 | 0
0 … 0 0 0
0 … 0 1 –1
0 … 0 0 0 | 0
0 … 0 0 0
0 … 0 0 1
0 … 0 0 1 | 1
0 … 0 0 1
0 … 0 0 0
0 … 0 0 0 | 0
0 … 0 1 –1

The graphs Y_{kkk}



MAPLE demonstrations

Y_{222}

Y_{333}

Y_{555}

a handy applet for manipulating small graphs

http://www.math.ucsd.edu/~llu/dgea