Two-graphs and Finite Geometries

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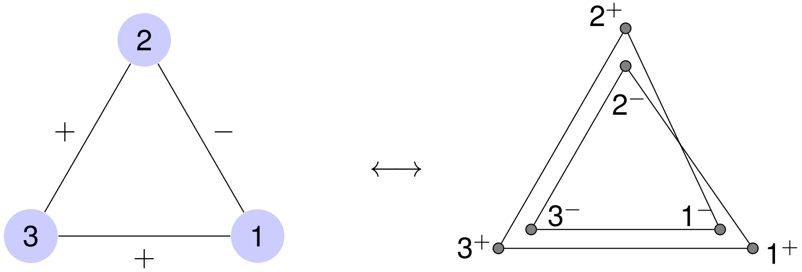
Double Covers of Triangles



K_3 with edges signed

double cover of K_3

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$$\sigma(1,2,3) := \sigma(1,2)\sigma(2,3)\sigma(3,1) = +1 \text{ or } -1.$$

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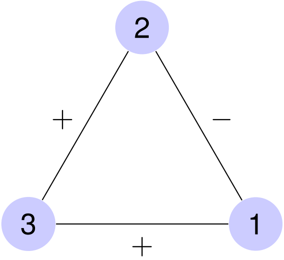

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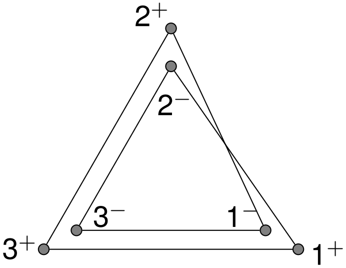

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Two-Graphs and Graphs

Let V be a set, and $\binom{V}{k}$ the collection of all k-subsets of V. A *two-graph on* V is a subset $\Delta \subseteq \binom{V}{3}$ such that every 4-subset $S \subseteq V$ contains an even number (i.e. 0, 2 or 4) triples in Δ.

The following three notions are equivalent:

$\left\{\begin{array}{c}\text{switching-equivalence}\\ \text{classes of graphs on } V\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{equivalence classes of double covers}\\ \text{of } K_V\text{, the complete graph on } V\end{array}\right\}$

$\left\{\text{two-graphs on } V\right\}$

Two-Graphs and Graphs

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$\{$switching-equivalence classes of graphs on $V\}$ $\longleftrightarrow$ $\{$equivalence classes of double covers of K_V, the complete graph on $V\}$

$\{$two-graphs on $V\}$

Two-graphs and Double Covers of Complete Graphs

Let V be any set, and consider an arbitrary signing of the complete graph K_V on V, i.e.

$$\sigma : \binom{V}{2} \to \{\pm 1\}.$$

Construct a graph K_V^σ on vertex set $V \times \{\pm 1\}$ with adjacency relation

$$x^\varepsilon \sim y^{\varepsilon'} \iff \varepsilon\varepsilon' = \sigma(x,y).$$

The covering map $K_V^\sigma \to K_V$ is determined by the vertex map $x^\varepsilon \mapsto x$.

The pairs $\{x,y\} \in \binom{V}{2}$ such that $\sigma(x,y) = -1$ form a graph on V which determines K_V^σ up to relabelling $i^+ \leftrightarrow i^-$. So K_V^σ corresponds to a switching-equivalence class of ordinary graphs on V.

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The triples $\{x,y,z\} \in \binom{V}{3}$ such that

$$\sigma(x,y,z) := \sigma(x,y)\sigma(y,z)\sigma(z,x) = -1$$

form a two-graph $\Delta = \Delta(\sigma)$ which corresponds to the double cover K_V^σ.

Let V be a finite-dimensional vector space over a finite field $\mathbb{F}_q$ of odd order. Let

$$B : V \times V \to \mathbb{F}_q$$

be a nondegenerate form on V (skew-symmetric, symmetric or hermitian).

A subspace $U \leqslant V$ is *totally isotropic* if $B(u, u') = 0$ for all $u, u' \in U$. All maximal totally isotropic subspaces have the same dimension n. The natural incidence structure formed by the totally isotropic subspaces of V is a polar space $\mathcal{P}$ of rank n.

Fix V, q, B, n, $\mathcal{P}$ as above.

Classical Polar Spaces

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Determinant Functions

Fix V, q, B, n, $\mathcal{P}$ as above.

Let $U \leqslant V$ be a totally isotropic (or totally singular) k-subspace, $k \leqslant n$. A *determinant function on U* is a nonzero k-linear map $\delta : U^k \to \mathbb{F}_q$ where $k = \dim U$, such that

$$\delta(u_{\tau(1)}, u_{\tau(2)}, \ldots, u_{\tau(k)}) = -\delta(u_1, u_2, \ldots, u_k)$$

for every *odd* permutation τ of $1, 2, \ldots, k$. Any such function δ forms a basis for the 1-space $(\bigwedge^k U)^* = Hom(\bigwedge^k U, \mathbb{F}_q)$.

To construct δ, one may first choose arbitrarily a basis $\psi_1, \psi_2, \ldots, \psi_k$ for $U^* = Hom(U, \mathbb{F}_q)$, then take

$$\delta(u_1, u_2, \ldots, u_k) = \det\left[\psi_i(u_j) : 1 \leqslant i, j \leqslant k\right].$$

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For each totally isotropic k-space $U \leqslant V$, choose a determinant function δ_U on U. Let $\chi : \mathbb{F}_q^\times \to \{\pm 1\}$ be the quadratic character.

For any two totally isotropic k-subspaces $U, U' \leqslant V$, define $\sigma(U, U') = \pm 1$ as follows. Choose bases $u_1, \ldots, u_k$ and $u'_1, \ldots, u'_k$ for U and U' respectively, such that $u_i = u'_i$ $(r < i \leqslant k)$ forms a basis for $U \cap U'$. Set

$$\begin{aligned}\sigma(U, U') &= \chi\Big(\delta_U(u_1, \ldots, u_k)\delta_{U'}(u'_1, \ldots, u'_k)\det[B(u_i, u'_j) : 1 \leqslant i, j \leqslant r]\Big)\\ &= \pm 1.\end{aligned}$$

Also for totally isotropic k-subspaces $U, U', U'' \leqslant V$, define

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Invariance of the Maslov Index

Theorem

Let $U, U' \leqslant V$ be totally singular k-subspaces.

(i) *The value of $\sigma(U, U')$ is independent of the choice of bases above.*

(ii) *Replacing δ_U by $c\delta_U$ results in replacing $\sigma(U, U')$ by $\chi(c)\sigma(U, U')$.*

(iii) *If B is symmetric or Hermitian, then $\sigma(U'.U) = \sigma(U, U')$. If B is alternating, then $\sigma(U', U) = (-1)^{k(q-1)/2}\sigma(U, U')$.*

(iv) *For every isometry g of B, $\sigma(U^g, U'^g) = \sigma(U, U')$.*

If B is alternating, assume $q \equiv 1 \mod 4$. Then the isometry group of B preserves $\sigma(U, U', U'')$ and the two-graph on $\mathfrak{U}_k$ (the collection of totally isotropic k-subspaces of V) given by

$$\Delta_k = \left\{\{U, U', U''\} \in \binom{\mathfrak{U}_k}{3} \,:\, \sigma(U, U', U'') = -1\right\}.$$

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Association Schemes from Symplectic Dual Polar Graphs

Now suppose B is alternating, $\mathcal{P}$ is a symplectic polar space, $2n = \dim V$, and assume $\frac{q-1}{2}n$ is even.

The vertex set of the symplectic dual polar graph is $\mathfrak{U}_n = \{\text{maximal totally isotropic subspaces}\}$. For $U, U' \in \mathfrak{U}_n$, the distance is

$$d(X,Y) = k \leqslant n \iff \dim(X/X\cap Y) = \dim(Y/X\cap Y) = k.$$

We construct a $(2n+1)$-class association scheme on $2|\mathfrak{U}_n| = 2q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)$ vertices X^+, X^- where $X \in \mathfrak{U}_n$. The relations ($k = 0, 1, 2, \ldots, n$) are

$$X^{\varepsilon} \overset{k}{\sim} Y^{\varepsilon'} \iff d(X,Y)=k,\ \varepsilon\varepsilon'=\sigma(X,Y);$$
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The $(2n+1)$-class association schemes constructed above are Q-polynomial (not P-polynomial). Can other such constructions be found, starting with a different polar space? or with different dimensions $k \in \{1, 2, \ldots, n-1\}$ of totally isotropic subspaces?

Gunawardena and M. (1995) used a two-graph argument to prove the nonexistence of ovoids in orthogonal polar spaces of type $O_{2n+1}(q)$, $n \geqslant 4$. Can the new invariants be used to solve other open problems regarding existence of ovoids, spreads or m-systems?

Questions

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References

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Thank You!

Questions?