p-Ranks

Lecture 5

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Zhejiang University—March 2019

p-Ranks of Finite Projective Planes

Let Π be a projective plane of order n with incidence matrix A.

Let p be a prime dividing n. (Only primes dividing n are of interest.)

The p-rank of Π is the rank of A over a field of characteristic p. This is an isomorphism invariant of Π (in fact, the easiest such invariant to compute).

Since $AA^T = nI + J$, we have the trivial upper bound $\text{rank}_p A \leqslant \frac{1}{2}\left(n^2+n+2\right)$ whenever $p \mid n$. Equality holds if $p = n$.

The best known lower bound is $\text{rank}_p A \geqslant n^{3/2}+1$ (Bruen and Ott, 1990; de Caen, Godsil and Royle, 1992).

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5-Ranks of Projective Planes of order 25

There are 99 known projective planes of order 25. Their 5-ranks are

$$226^1,\ 239^1,\ 251^1,\ 253^1,\ 255^1,\ 256^1,\ 257^1,\ 258^3,\ 259^3,\ 260^2,\ 261^2,\ 262^5,\ 264^2,\ 266^1,\ 268^3,\ 269^1,\ 271^1,\ 272^2,\ 273^1,\ 274^3,\ 275^4,\ 276^6,\ 277^6,\ 278^{12},\ 279^{27},\ 280^6,\ 286^1,\ 300^1$$

where r^k indicates k planes of rank r.

The plane with smallest 5-rank is the classical plane $\mathbb{P}^2\mathbb{F}_{25}$. The largest 5-rank occurs for a derived Hughes plane.

Computation of p-rank is difficult for large matrices not because of execution time, but due to limits on available RAM.

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Open Questions

Q: Does $\mathbb{P}^2\mathbb{F}_q$ have the smallest p-rank among all projective planes of order $q = p^e$? (The **Hamada-Sachar Conjecture**).

Q: Improve the upper and lower bounds for $\text{rank}_p A$ in general. For $n = 25$ we know $126 \leqslant \text{rank}_p A \leqslant 326$, but all known planes have p-rank in the interval $[226, 300]$.

Q: Improve the known upper bound for p-ranks of translation planes (Key and MacKenzie, 1991). For $q = 25$ this upper bound is 296; the translation planes have rank $\leqslant 264$.

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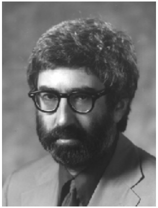
p-Ranks and Related/Current Work

The study of p-ranks of incidence matrices extends naturally to questions about Smith Normal Forms and decomposition of the associated $\mathbb{F}_p$-codes as $\mathbb{F}_pG$-modules.



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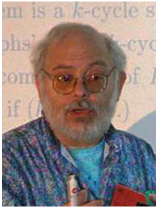

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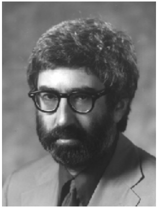
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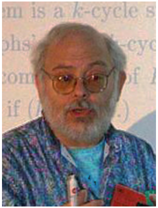
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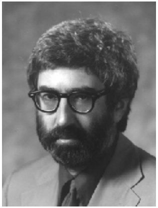

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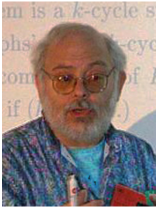

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Points versus Hyperplanes in Projective Space

Let A be the incidence matrix of points versus hyperplanes in $\mathbb{P}^n\mathbb{F}_q$, $q{=}p^e$. Then

$$\operatorname{rank}_p A = \binom{p+n-1}{n}^e + 1.$$

Theorem (Blokhuis and M., 1995)

If $p^{\lfloor n/2 \rfloor} > \binom{p+n-1}{n}$, then quadrics in $\mathbb{P}^n\mathbb{F}_q$ contain no ovoids.

In particular, there are no ovoids in quadrics in $\mathbb{P}^9\mathbb{F}_{2^e}$, $\mathbb{P}^9\mathbb{F}_{3^e}$, $\mathbb{P}^{11}\mathbb{F}_{5^e}$, $\mathbb{P}^{11}\mathbb{F}_{7^e}$, etc.

Proof. If $\mathcal{O} = \{P_1, P_2, \ldots, P_m\}$ is an ovoid, then the points of $\mathcal{O}$ and the hyperplanes $P_1^\perp, \ldots, P_m^\perp$ index the rows and columns of an identity submatrix I_m in A. Comparing p-ranks,

$$m = p^{\lfloor n/2 \rfloor e}+1 \leqslant \binom{p+n-1}{n}^e+1. \qquad \square$$

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An Improvement

A quadric partitions the point-hyperplane incidence matrix of $\mathbb{P}^n\mathbb{F}_q$ as

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{12}^T & A_{22} \end{array}\right]$$

where rows (and columns) of A_{11} are indexed by points of the quadric (and tangent hyperplanes).

Sharper bounds for ovoids follow from

$$|\mathcal{O}| = m \leqslant \operatorname{rank}_p A_{11} \leqslant \operatorname{rank}_p[A_{11}|A_{12}] \leqslant \operatorname{rank}_p A.$$

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$$rank_p\left[A_{11}|A_{12}\right] = \left[\binom{p+n-1}{n} - \binom{p+n-3}{n}\right]^e + 1.$$

So there are no ovoids in quadrics if $p^{\lfloor n/2 \rfloor} > \binom{p+n-1}{n} - \binom{p+n-3}{n}$.

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Instances when the p-Rank Bound is Tight

Ovoids in Triality Quadrics of $\mathbb{P}^7\mathbb{F}_q$, $q = 2^e$

Ovoids have size $|\mathcal{O}| = q^3 + 1 = \operatorname{rank}_2 A_{11}$.

Only known examples: Two infinite families (admitting $PSL_3(q)$, all e; and $PSU_3(q)$, e odd; and one sporadic example, $q{=}8$).

Ovoids in Parabolic Quadrics of $\mathbb{P}^6\mathbb{F}_q$, $q = 3^e$

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Ovoids of $\mathbb{P}^3\mathbb{F}_q$, $q = 2^e$. ($q^2{+}1$ points, no three points collinear)

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Points versus k-subspaces of $\mathbb{P}^n\mathbb{F}_q$

Let A be the incidence matrix of points versus k-subspaces of $\mathbb{P}^n\mathbb{F}_q$, $q = p^e$. Let M be the $k \times k$ matrix whose (i,j)-entry equals the coefficient of t^{pi-j} in $(1+t+t^2+\cdots+t^{p-1})^{n+1}$. Then

$$\mathrm{rank}_p A = 1 + \left(\text{coefficient of } t^e \text{ in } tr\left[(I-tM)^{-1}\right]\right).$$

Example: Points versus Lines of $\mathbb{P}^3\mathbb{F}_{5^e}$

$(1+t+\cdots+t^4)^3 = 1+4t+10t^2+20t^3+35t^4+\cdots+85t^8+80t^9+\cdots$
so $M = \begin{bmatrix} 35 & 80 \\ 20 & 85 \end{bmatrix}$ and

$$tr\left[(I-tM)^{-1}\right] = \frac{2(1-60t)}{1-120t+1375t^2}$$
$$=2+120t+11650t^2+1233000t^3+131941250t^4+14137575000t^5+\cdots$$

so $\mathrm{rank}_5 A = 121, 11651$, etc. for $q = 5, 25, \ldots$.

Hamada's Formula (1968) also expresses $\mathrm{rank}_p A$ as a multiple sum, requiring exponential time to evaluate.

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$$=2+120t+11650t^2+1233000t^3+131941250t^4+14137575000t^5+\cdots$$

so $\operatorname{rank}_5 A = 121, 11651$, etc. for $q = 5, 25, \ldots$.

Hamada's Formula (1968) also expresses $\operatorname{rank}_p A$ as a multiple sum, requiring exponential time to evaluate.

Points versus k-subspaces of $\mathbb{P}^n\mathbb{F}_q$

Let A be the incidence matrix of points versus k-subspaces of $\mathbb{P}^n\mathbb{F}_q$, $q = p^e$. Let M be the $k \times k$ matrix whose (i, j)-entry equals the coefficient of t^{pi-j} in $(1 + t + t^2 + \cdots + t^{p-1})^{n+1}$. Then

$$\text{rank}_p A = 1 + \left(\text{coefficient of } t^e \text{ in } tr\left[(I - tM)^{-1}\right]\right).$$

Example: Points versus Lines of $\mathbb{P}^3\mathbb{F}_{5^e}$

$(1+t+\cdots+t^4)^3 = 1+4t+10t^2+20t^3+35t^4+\cdots+85t^8+80t^9+\cdots$

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p-Ranks of Algebraic Sets of Points vs. Hyperplanes

Choose homogeneous coordinates $x_0, x_1, \ldots, x_n$ for $\mathbb{P}^n F$, $F = \mathbb{F}_q$, $q = p^e$. The polynomial ring $R = F[x_0, x_1, \ldots, x_n]$ is graded by degree:

$$R = \bigoplus_{k \geqslant 0} R_k$$

where R_k consists of k-homogeneous polynomials in $x_0, x_1, \ldots, x_n$. Let $\mathfrak{I} \subseteq R$ be a homogeneous ideal, i.e. $\mathfrak{I}$ is generated by a set of homogeneous polynomials. The points of $\mathbb{P}^n F$ where all $f \in \mathfrak{I}$ vanish is an algebraic point set $\mathcal{Z}(\mathfrak{I})$. We want to know $\text{rank}_p A_{\mathfrak{I}}$ where

$$A = \begin{bmatrix} A_1 = A_{\mathfrak{I}} \\ \hline \\ A_2 \\ \\ \end{bmatrix} \begin{matrix} \} \mathcal{Z}(\mathfrak{I}) \\ \\ \\ \\ \end{matrix} ;$$

here rows and columns of $A_{\mathfrak{I}}$ are indexed by points of $\mathcal{Z}(\mathfrak{I})$, and *all* hyperplanes of $\mathbb{P}^n F$.

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Hilbert Functions

The homogeneous ideal $\mathfrak{I} = \bigoplus_{k\geqslant 0}\mathfrak{I}_k$ where $\mathfrak{I}_k = \mathfrak{I}\cap R_k$ also has a grading quotient ring

$$R/\mathfrak{I} = \bigoplus_{k\geqslant 0}\big(R_k/\mathfrak{I}_k\big).$$

The Hilbert function of $\mathfrak{I}$ is $h_{\mathfrak{I}}(k) = \dim\big(R_k/\mathfrak{I}_k\big)$. The generating function for its sequence of values is the Hilbert series

$$\mathrm{Hilb}_{\mathfrak{I}}(t) = \sum_{k\geqslant 0} h_{\mathfrak{I}}(k)t^k$$

which is actually a rational function $\mathrm{Hilb}_{\mathfrak{I}}(t)\in\mathbb{Q}(t)$. That is, for $k\gg 0$, $h_{\mathfrak{I}}(k)$ coincides with a polynomial. This is the Hilbert polynomial of $\mathfrak{I}$, whose leading term $m\frac{k^d}{d!}$ defines the degree m and dimension d of $\mathcal{Z}(\mathfrak{I})$.

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Example: Projective n-space $\mathbb{P}^n F$

$\mathfrak{I} = (0)$ has zero set $\mathcal{Z}((0)) = \mathbb{P}^n F$ with Hilbert function

$$h_{(0)}(k) = \dim\bigl(R_k/(0)\bigr) = \dim R_k = \tbinom{k+n}{n}$$
$$= \tfrac{1}{n!}(k{+}1)(k{+}2)\cdots(k{+}n).$$

The leading term $\frac{k^n}{n!}$ tells us that $\mathbb{P}^n F$ has dimension n and degree 1.

The Hilbert series is

$$\mathrm{Hilb}_{(0)}(t) = \sum_{k\geqslant 0} \binom{k+n}{n} t^k = \frac{1}{(1-t)^{n+1}}.$$

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Example: Quadrics in $\mathbb{P}^n F$

A quadric $\mathcal{Z}(Q)$ is the zero set of a homogeneous quadratic polynomial $Q(x_0, x_1, \ldots, x_n) \in R_2$.

$\mathfrak{I} = (Q)$ and $\mathfrak{I}_k = QR_{k-2}$ for $k \geqslant 2$. ($\mathfrak{I}_0 = \mathfrak{I}_1 = 0$.)

The Hilbert function is

$$h_{(Q)}(k) = \begin{cases} 0, & \text{for } k = 0, 1; \\ \binom{k+n}{n} - \binom{k+n-2}{n}, & \text{for } k \geqslant 2. \end{cases}$$

The leading term $2\frac{k^{n-1}}{(n-1)!}$ tells us that the quadric has dimension $n-1$ and degree 2.

The value $h_{(Q)}(p-1) = \binom{p+n-1}{n} - \binom{p+n-3}{n}$ gives $\operatorname{rank}_p A_{(Q)} = 1 + h_{(Q)}(p-1)$ over the prime field $F = \mathbb{F}_p$.

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Other Algebraic Sets

We have also computed $\operatorname{rank}_p A_{\mathfrak{I}}$ for several other algebraic sets $\mathcal{Z}(\mathfrak{I})$, including hermitian varieties and Grassmann varieties.

In particular, we get bounds for ovoids in other finite classical polar spaces.

Disclaimer: In general, the ideal $\mathfrak{I} \subseteq R$ needs to be replaced by a slightly larger ideal:

$$\mathfrak{I} \subseteq \widehat{\mathfrak{I}} = \sqrt{\mathfrak{I} + J} \subseteq R$$

where $J = (x_i^q x_j - x_i x_j^q \; : \; i, j)$.

Here $\widehat{\mathfrak{I}}$ is the set of all $f \in R$ vanishing on $\mathcal{Z}(\mathfrak{I})$.

In place of $h_{\mathfrak{I}}(p-1)$ we should really have $h_{\widehat{\mathfrak{I}}}(p-1)$; but in many settings, we show that these two values agree.

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Classical Generalized Quadrangles

A classical generalized quadrangle of order (q, q) has $q+1$ points on each line and $q+1$ lines through each point, $q = p^e$. Let A be its incidence matrix.

Theorem (Sastry and Sin, 1996; de Caen and M., 2000; Chandler, Sin and Xiang, 2006)

For $q = 2^e$, $\operatorname{rank}_p A = 1 + \left(\frac{1+\sqrt{17}}{2}\right)^{2e} + \left(\frac{1-\sqrt{17}}{2}\right)^{2e}$.

For $q = p$, $\operatorname{rank}_p A = 1 + \frac{p(p+1)^2}{2}$.

For $q = p^e$, *p odd*, $\operatorname{rank}_p A = 1 + \alpha_+^e + \alpha_-^e$ *where*

$$\alpha_\pm = \frac{p(p+1)^2}{4} \pm \frac{p(p^2-1)}{12}\sqrt{17}.$$

Classical Generalized Quadrangles

A classical generalized quadrangle of order (q, q) has $q+1$ points on each line and $q+1$ lines through each point, $q = p^e$. Let A be its incidence matrix.

Theorem (Sastry and Sin, 1996; de Caen and M., 2000; Chandler, Sin and Xiang, 2006)

For $q = 2^e$, $\operatorname{rank}_p A = 1 + \left(\frac{1+\sqrt{17}}{2}\right)^{2e} + \left(\frac{1-\sqrt{17}}{2}\right)^{2e}$.

For $q = p$, $\operatorname{rank}_p A = 1 + \frac{p(p+1)^2}{2}$.

For $q = p^e$, *p odd*, $\operatorname{rank}_p A = 1 + \alpha_+^e + \alpha_-^e$ *where*

$$\alpha_\pm = \frac{p(p+1)^2}{4} \pm \frac{p(p^2-1)}{12}\sqrt{17}.$$

Other Generalized Quadrangles of order (q, q)?

The only known generalized quadrangles of order (q, q) are the classical ones from $Sp(4, q)$ and $O(5, q)$.

Let p be prime. Any generalized quadrangle of order (n, n) has $\text{rank}_p A \geqslant n^2 + 1$ (de Caen, Godsil and Royle, 1992).

The classical GQ of order $(5, 5)$ has p-rank equal to 91. The lower bound is 26.

Q: Improve the lower bound for p-ranks of GQ's of order (q, q).

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