Ovoids and Spreads

Lecture 4

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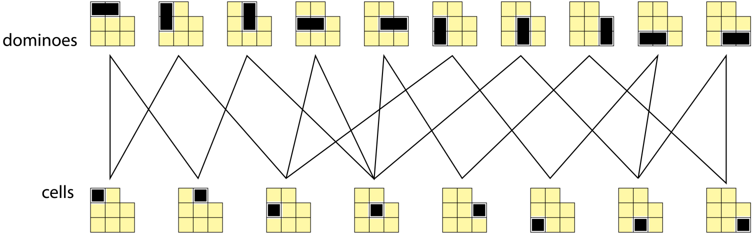
University of Wyoming

Zhejiang University—March 2019

Consider a bipartite graph representing incidences between *points* and *blocks*.



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Dually, an ovoid is a set of points partitioning the blocks.

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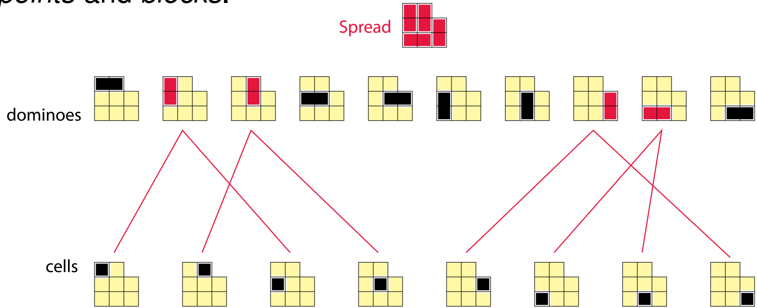

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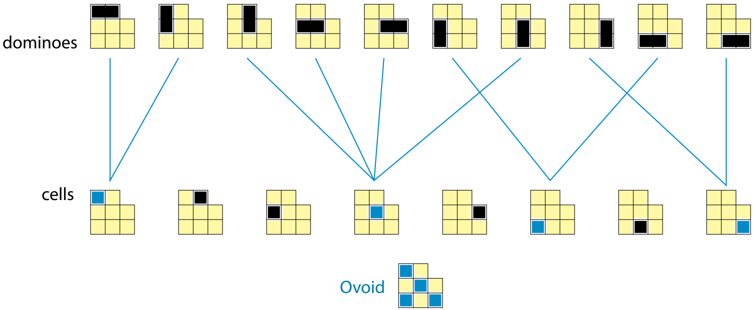

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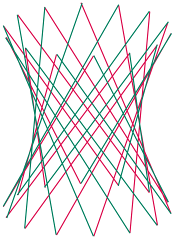
Hyperbolic Quadrics in $\mathbb{P}^3\mathbb{F}_q$

A *hyperbolic quadric* (ruled quadric) in projective 3-space is combinatorially just a $(q+1) \times (q+1)$ grid. It has $(q+1)^2$ points and $2(q+1)$ lines.



Here is one spread and here is the other spread.

There are $(q+1)!$ ovoids (transversals of the grid).

This includes $q(q^2-1)$ regular ovoids, which are nonsingular plane sections of the quadric.

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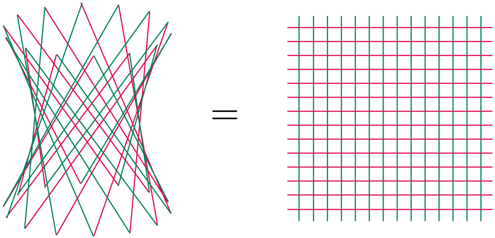

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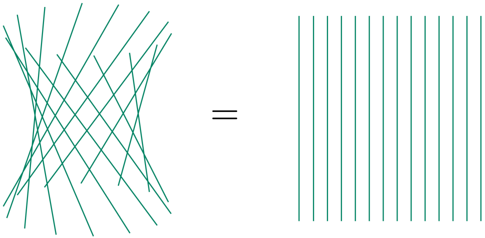

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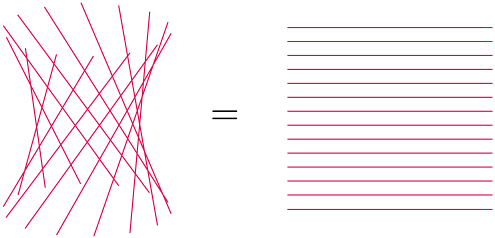

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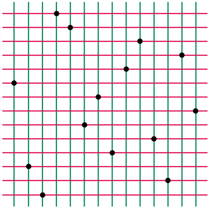

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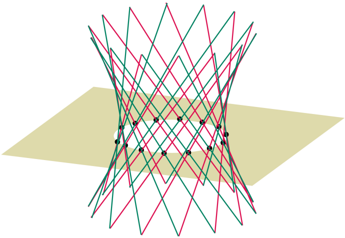

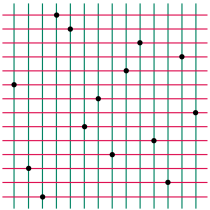

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Some ovoids in the Klein quadric in $\mathbb{P}^5\mathbb{F}_p$

Consider a prime $p \equiv 1 \bmod 4$. Let $\mathcal{S}$ be the set of all $x = (x_1, \ldots, x_6) \in \mathbb{Z}^6$ such that

1. $x_i \equiv 1 \bmod 4$; and
2. $\sum_i x_i^2 = 6p$.

Then $|\mathcal{S}| = p^2 + 1$; and for all $x \neq y$ in $\mathcal{S}$, $x \cdot y \not\equiv 0 \bmod p$.

Example ($p = 5$, $|\mathcal{S}| = 5^2 + 1 = 26$)

$\mathcal{S}$ contains 6 vectors of shape $(5, 1, 1, 1, 1, 1)$;
20 vectors of shape $(-3, -3, -3, 1, 1, 1)$.

Example ($p = 13$, $|\mathcal{S}| = 13^2 + 1 = 170$)

$\mathcal{S}$ contains 20 vectors of shape $(5, 5, 5, 1, 1, 1)$;
30 vectors of shape $(-7, -5, 1, 1, 1, 1)$;
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Points of $\mathbb{P}^5\mathbb{F}_p$ satisfying $\sum_i x_i^2 = 0$ form the Klein quadric with $(p^2{+}1)(p^2{+}p{+}1)$ points and $2(p^2{+}1)(p{+}1)$ planes. Each point is in $2(p{+}1)$ planes, so an *ovoid* is any set of $p^2{+}1$ points, no two on the same plane (no two perpendicular).

This is the same as a spread of lines in $\mathbb{P}^3\mathbb{F}_p$, i.e. a partition of the $(p^2{+}1)(p{+}1)$ points into $p^2{+}1$ lines (of size $p{+}1$).

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S_6-invariant ovoids in the Klein quadric

Now let $F = \mathbb{F}_q$, $q \equiv 1 \mod 4$. The vectors in F^6 satisfying $\sum_i x_i^2 = 0$ form (projectively) a Klein quadric.

When can we find an ovoid in the quadric invariant under S_6 acting by coordinate permutations? (We need q^2+1 vectors satisfying $x \cdot y = 0$ iff $x = y$. And we want the set to be invariant under coordinate permutations.)

Q: Must q be prime?

Usually when a combinatorial problem has a solution over $\mathbb{F}_p$, the solution generalises to $\mathbb{F}_q$, $q = p^e$. The situation above seems to be an example to the contrary.

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The E_8 Root Lattice

Let E be the set of all vectors $\frac{1}{2}(x_1, x_2, \ldots, x_8) \in \mathbb{Q}^8$ such that

$$x_i \in \mathbb{Z}, \quad x_1 \equiv x_2 \equiv \cdots \equiv x_8 \mod 2, \quad \text{and } \sum_i x_i \equiv 0 \mod 4.$$

This is the E_8 root lattice. It is

- a lattice (i.e. additive subgroup of $\mathbb{R}^8$);
- integral ($x \cdot y \in \mathbb{Z}$ for all $x, y \in E$);
- unimodular (its density is 1, i.e. it has one point per unit volume on average);
- it has minimum distance $\sqrt{2}$ (so for any $x \neq y$ in E, $\|y - x\| \geqslant \sqrt{2}$); and
- it is unique with these properties. Any subset of $\mathbb{R}^8$ of density 1 has minimum distance at most $\sqrt{2}$; and up to isometry, E is the unique subset attaining this optimum.

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E has 240 shortest vectors ($e \in E$, $\|e\|^2 = e \cdot e = 2$) called root vectors:

- $(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$ and permutations thereof (112 vectors of this shape); and
- $\frac{1}{2}(\pm 1, \pm 1, \ldots, \pm 1)$ with an even number of '$-$' signs (128 vectors of this shape).

For an odd prime p, there are $240(p^3+1)$ vectors $x \in E$ with $\|x\|^2 = 2p$.

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The Triality Quadric in $\mathbb{P}^7\mathbb{F}_q$

Let $F = \mathbb{F}_q$ where q is an odd prime. The triality quadric in $\mathbb{P}^7 F$ with equation $\sum_i x_i^2 = 0$ contains

- $(q^3+1)(q^2+1)(q+1)$ points;
- $2(q^3+1)(q^2+1)(q+1)$ solids, i.e. projective 3-spaces, the maximum dimension of any subspaces lying in the quadric; and
- $2(q^2+1)(q+1)$ solids containing each point.

So an *ovoid* (set of points hitting each solid exactly once) must have size q^3+1.

But do ovoids *exist* in the triality quadric?

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Let $F = \mathbb{F}_q$ where q is an odd prime. The triality quadric in $\mathbb{P}^7 F$ with equation $\sum_i x_i^2 = 0$ contains

- $(q^3+1)(q^2+1)(q+1)$ points;
- $2(q^3+1)(q^2+1)(q+1)$ solids, i.e. projective 3-spaces, the maximum dimension of any subspaces lying in the quadric; and
- $2(q^2+1)(q+1)$ solids containing each point.

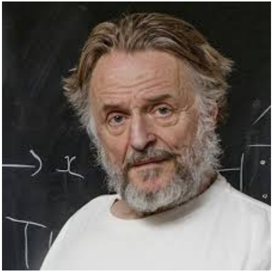
So an *ovoid* (set of points hitting each solid exactly once) must have size q^3+1.

But do ovoids *exist* in the triality quadric?

Theorem (Conway et. al., 1988)

For every prime p, there is an ovoid in the triality quadric in $\mathbb{P}^7\mathbb{F}_p$.

Take p to be an *odd* prime (the case $p = 2$ was previously solved). Fix a root vector $e \in E$. Let $\mathcal{S}$ be the set of all $v \in E$ such that $\|v\|^2 = 2p$ and $v = e + 2x$ for some $x \in E$. We easily conclude that $|\mathcal{S}| = 2(p^3+1)$ and $\mathcal{S}$ consists of p^3+1 pairs $\pm v$ which reduce (mod p) to give an ovoid in the triality quadric. □



John H. Conway

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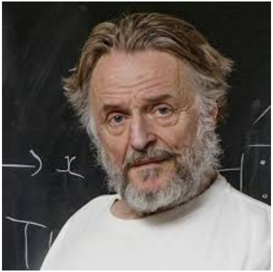

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Conway's Ovoids

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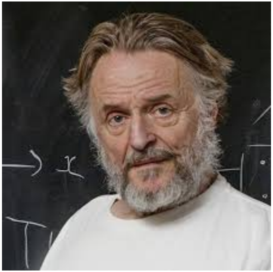

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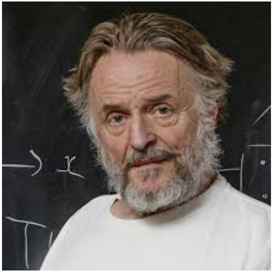

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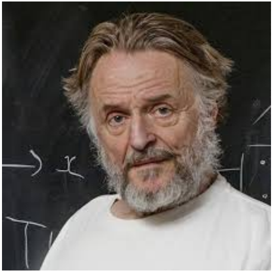

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More: Ovoids in the Triality Quadric

Using E, Conway et. al. gave more examples of ovoids in the triality quadric (up to 3 examples for each prime p).

Also using E, we (1993) generalized this to an unbounded number of examples for each p. Other constructions of ovoids in the triality quadric are known, but *almost all of them come from the E_8 root lattice.*

It is really this construction of ovoids from E_8, which explains the earlier examples of ovoids in the Klein quadric (and *many* similar examples).

The geometry of the triality quadric admits a triality automorphism, mapping ovoids to spreads.

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Open Questions

Q: Do ovoids exist in the triality quadric in $\mathbb{P}^7\mathbb{F}_q$ for every q? The smallest open case is $q = 25$.

Q: Is it true that the number of ovoids in the triality quadric is unbounded as $p \to \infty$?

Q: If an ovoid in the triality quadric admits certain groups (such as $Sp_6(\mathbb{F}_q)$), must it come from the E_8 construction, with $q = p$?

Q: Construct an ovoid in the triality quadric admitting no automorphisms (a 'rigid' ovoid).

Q: Prove that for the triality quadric in $\mathbb{P}^7\mathbb{F}_p$ arising from E_8, the total number of ovoids arising from E_8 is $\frac{|G(p)|}{4|G(2)|}(p^4 + 239)$ where $G = PGO_8^+(p)$.

Q: Are there any ovoids in quadrics in $\mathbb{P}^k\mathbb{F}_q$, $k > 7$?

Q: The Leech lattice mod p does not give ovoids but . . . what *does* it give?

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Fans

A fan of a quadric (or any finite classical polar space) is a partition of its points into ovoids. E.g. $(q^2+1)(q+1)$ ovoids of size q^3+1 in the triality quadric.

Very little has been written on the subject, only a few examples and a few nonexistence results, over the past 30 years. This should change, now that Cameron has showed us why the existence question for fans (also ovoids and spreads) is forced upon us in permutation group theory:

Theorem

A classical group is non-synchronizing if and only if its polar space possesses either

- *an ovoid and a spread, or*
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Q: When do fans exist?

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