Geometry Beyond the Finite

Lecture 3

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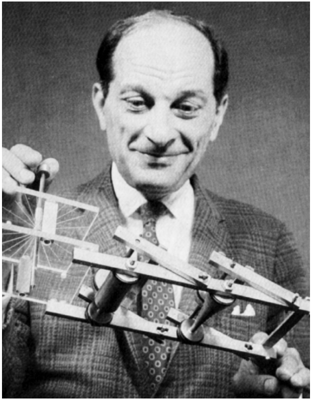

“The infinite case we shall do right away.
The finite may take a little longer.”

—Stanisław Ulam

(Partial) Spreads

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space.

A partial spread is a collection of mutually disjoint lines $\Sigma \subseteq \mathcal{L}$, i.e. no two lines in Σ intersect. A spread is a partial spread covering the points, i.e. $\bigcup \Sigma = \mathcal{P}$. Equivalently, a spread is a partition of the points into lines.

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Projective 3-space over a field F is denoted $\mathbb{P}^3 F$. It has points, lines and planes given by the subspaces of F^4 of dimension 1, 2 and 3.

In the finite case, $\mathbb{P}^3\mathbb{F}_q$ has $(q^2+1)(q+1)$ points, and each line has $q+1$ points. So every partial spread has at most q^2+1 lines. A spread is the same thing as a set of q^2+1 mutually disjoint lines. Every such spread gives a plane of order q^2, known as a translation plane. Take F^4 as points, and the *cosets* of the q^2+1 subspaces $\ell \in \Sigma$ as lines, to get an affine plane of order q^2; and this gives the affine translation plane arising from Σ.

The biggest obstacle to constructing spreads (and hence translation planes) is the fact that not every partial spread can be extended to a spread. But the infinite case is much easier:

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Line Spreads in $\mathbb{P}^3 F$

Let F be an infinite field. Then every partial spread of $\mathbb{P}^3 F$ having fewer than $|F|$ lines, can be extended to a spread, by a process of transfinite induction.

(In order to get a translation plane, we need the spread to also be a dual spread: every plane of $\mathbb{P}^3 F$ should contain a line of the spread. But this is easily arranged. In the finite case every spread is also a dual spread, by the pigeonhole principle, so this issue doesn't even arise.)

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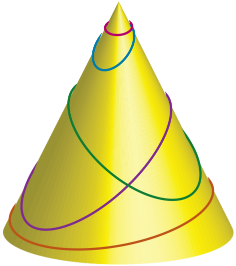
Extending Partial Flocks to Flocks

I used a similar idea to extend a partial flock to a flock of a quadratic cone, answering a question posed by Norman Johnson at the 1996 conference *Mostly Finite Geometries*.

A flock of a quadratic cone is a partition of its $q(q+1)$ points (minus the vertex) into q conics of size $q+1$.



Collineations of Projective Planes

A collineation of a projective plane Π is a bijection of its points, which also gives a bijection on the lines. An automorphism of the bipartite incidence graph is either a *collineation* (mapping points $\to$ points, and lines $\to$ lines) or a correlation (interchanging points $\leftrightarrow$ lines).

The group of all collineations of Π is denoted $\mathrm{Aut}\,\Pi$.

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Let Π be a projective plane.

In the finite case, Aut Π has equally many orbits on points and on lines. (The same conclusion holds for automorphisms of symmetric designs, character tables of finite groups, etc.)

Does the same conclusion hold for infinite projective planes Π? Cameron (1991) posed this question, which he attributed to Kantor. We found a negative answer to this question:

Theorem (M. and Penttila, 2014)

Let A and B be any two nonzero cardinal numbers. Then there is a projective plane whose collineation group has A orbits on points and B orbits on lines.

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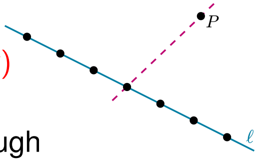
Generalized Quadrangles



A GQ (generalized quadrangle) of order (s, t) has

- $s+1$ points on every line; $t+1$ lines through every point;
- if point P is not on line ℓ, then there is a unique line through P meeting ℓ.

We generally require $s, t > 1$.

Finite case ($s, t < \infty$): There are $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines. Also $\sqrt{t} \leqslant s \leqslant t^2$, i.e. $\sqrt{s} \leqslant t \leqslant s^2$.

Q: Can one of s, t be finite and the other infinite?

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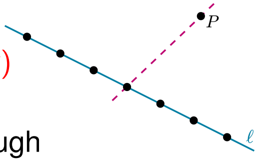

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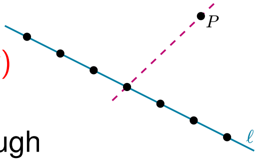

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Semifinite Generalized Quadrangles

Suppose a GQ has finite linesize $s{+}1 < \infty$. Must the GQ be finite (i.e. is $t < \infty$)?

A GQ with three points per line is finite. (Elementary proof, one short paragraph; Cameron)

A GQ with four points per line is finite. (Four-page paper; Brouwer 1991)

A GQ with five points per line is finite. (Cherlin 2005 using mathematical logic)

Line size six (and higher) is completely open!

Mathematical logic supplies methods for *constructing examples* but also for *proving nonexistence*.

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Orbits of Aut Π on k-tuples of points of $\Pi = \mathbb{P}^2 F$

Consider the classical projective plane $\Pi = \mathbb{P}^2 F$ over a field F.

Aut Π permutes points transitively (i.e. just one orbit on points).

There are two orbits on ordered pairs of points: (P, P), (P, Q), $P \neq Q$.

There are six orbits on ordered triples of points: (P, P, P), (P, P, Q), (P, Q, P), (Q, P, P), (P, Q, R) collinear, (P, Q, R) noncollinear where P, Q, R are distinct.

If F is infinite, there are *infinitely many orbits* on 4-tuples of points. (For four collinear points, the cross ratio has infinitely many possible values.)

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Let Π be a (countably) infinite plane.

Suppose that for every $k \geqslant 1$, Aut Π has only *finitely many orbits* on k-tuples of points. (One could say instead k-sets of points, or k-sets of lines, etc. and this condition is unchanged.)

Then we say Π is $\aleph_0$-categorical.

Open Problem

Does there exist an $\aleph_0$-categorical plane?

As we have indicated, an $\aleph_0$-categorical plane *cannot be classical*.

If there exists an $\aleph_0$-categorical plane, it would have a *much* bigger group of automorphisms (more transitive and larger cardinality) than any classical plane. This seems inconceivable.

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Suppose Π is an $\aleph_0$-categorical projective plane. We try to get a contradiction.

Lemma

Any finite list of points $P_1, P_2, \ldots, P_r$ which includes a quadrangle, must generate a finite subplane.

Proof. Starting with $\mathcal{P}_1 = \{P_1, P_2, \ldots, P_r\}$ and alternately joining points and intersecting lines, we get a sequence of substructures with point sets $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3 \subseteq \cdots$ whose union gives a subplane $\mathcal{P}_\infty = \bigcup_i \mathcal{P}_i \subseteq \Pi$.

If this sequence is *strictly increasing*, choose points $T_i \in \mathcal{P}_{i+1} \smallsetminus \mathcal{P}_i$; then the $(r+1)$-tuples $(P_1, P_2, \ldots, P_r, T_i)$ $(i \geqslant 1)$ are in distinct orbits, a contradiction. So we must have $\mathcal{P}_i = \mathcal{P}_\infty$ for all sufficiently large i. □

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Now let (P_i, Q_i, R_i, S_i) $(i = 1, 2, \ldots, m)$ be representatives of the orbits of $\mathrm{Aut}\,\Pi$ on quadrangles. If n_i is the order of the subplane generated by P_i, Q_i, R_i, S_i, then *every quadrangle* generates a subplane of order at most

$$n = \max\{n_1, n_2, \ldots, n_m\}.$$

Let $\Pi_0 \subset \Pi$ be any finite subplane of order $> n$. (This is easily found; for example any quadrangle, together with any $n+2$ collinear points, will together generate such a subplane.) Note that every quadrangle in Π_0 generates a proper subplane.

Without loss of generality, Π_0 is non-classical. (As we have observed, Π itself is non-classical. So there exists an induced substructure in Π violating Desargues' Theorem. Without loss of generality, Π_0 was chosen so as to contain this substructure.)

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Without loss of generality, Π_0 is non-classical. (As we have observed, Π itself is non-classical. So there exists an induced substructure in Π violating Desargues' Theorem. Without loss of generality, Π_0 was chosen so as to contain this substructure.)

$\aleph_0$-categorical projective planes

Now let (P_i, Q_i, R_i, S_i) $(i = 1, 2, \ldots, m)$ be representatives of the orbits of $\mathrm{Aut}\,\Pi$ on quadrangles. If n_i is the order of the subplane generated by P_i, Q_i, R_i, S_i, then *every quadrangle* generates a subplane of order at most

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So if there exists an $\aleph_0$-categorical plane, then we obtain *many* examples of finite nonclassical planes in which every quadrangle generates a proper subplane.

Such a conclusion seems highly improbable, although we have not yet reached a contradiction.

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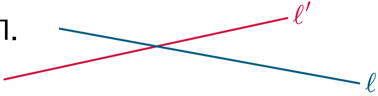
Perspectivities and Projectivities

Fix lines $\ell \neq \ell'$ in a projective plane Π.



Each point $O \notin \ell \cup \ell'$ determines a bijection from the points of ℓ to the points of ℓ', called a perspectivity.

Compositions of perspectivities gives the projectivity groupoid of Π. In particular for each line ℓ, we get a group of permutations of the points of ℓ called the projectivity group of Π. This group doesn't really depend on the choice of line ℓ.

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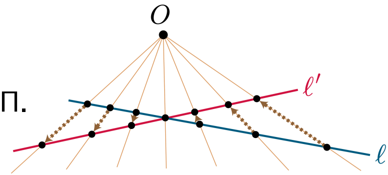

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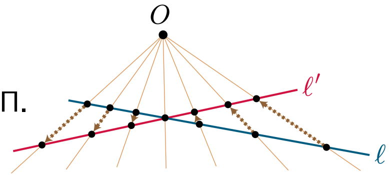

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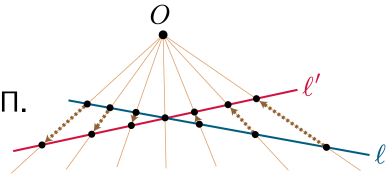

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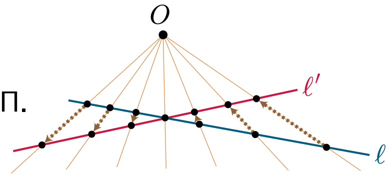

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Perspectivities and Projectivities

A classical plane $\Pi = \mathbb{P}^2 F$ has collineation group $\mathrm{Aut}\,\Pi \cong P\Gamma L_3(F)$ and projectivity group $PGL_2(F)$.

A finite nonclassical plane Π of order n has typically *small* collineation group $\mathrm{Aut}\,\Pi$, but *very large* projectivity group A_{n+1} or S_{n+1}.

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