Planes and their Substructures

Lecture 2

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In this talk, all planes considered are *projective*.

Every field F gives rise to a classical projective plane $\mathbb{P}^2 F$ whose points and lines are the one- and two-dimensional subspaces of F^3.

A projective plane of order $n \geqslant 2$ has n^2+n+1 points and n^2+n+1 lines.
Each line has $n+1$ points, and each point is on $n+1$ lines.
Any two points lie on exactly one common line.
Any two distinct lines meet in exactly one point.

The finite classical plane $\mathbb{P}^2\mathbb{F}_q$ has prime power order q. And there exist finite non-classical planes, but in all known cases, the order is a prime power. And the only known planes of prime order are the classical ones $\mathbb{P}^2\mathbb{F}_p$.

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(Partial) Linear Spaces

A partial linear space (PLS) is an incidence system of points and lines in which no two distinct points are joined by more than one line, i.e.  *does not* occur. (Some authors require every line to have at least two points. Connectedness is never required.)

The point-line dual of a partial linear space (in which we reverse the roles of points and lines) is a partial linear space. The incidence graph of a PLS is a bipartite graph of girth at least 6 (i.e. no 4-cycles).

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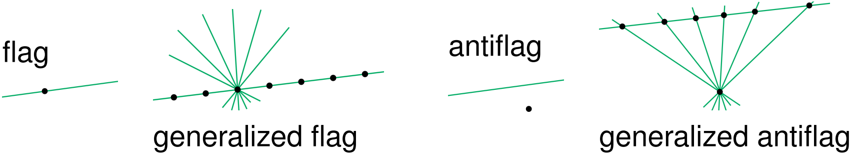
Linear Spaces and Projective Planes

If a linear space is also a dual linear space (i.e. any two lines intersect), then either

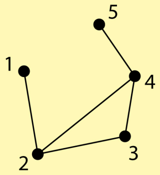
- It contains a quadrangle (four points, no three collinear). In this case the space is a projective plane. *Or*
- It does not contain a quadrangle. In this case it is a generalized flag or a generalized antiflag.



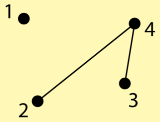
Subgraphs and Induced Subgraphs

Let Γ be a graph with vertex set V. An induced subgraph of Γ has vertex set $V' \subseteq V$. Its edges are all edges of Γ joining vertices in V'.

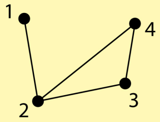
A subgraph of Γ has vertex set $V' \subseteq V$, and edges given by a *subset* of the edges in Γ joining vertices in V'.



a graph Γ



a *subgraph* of Γ (*not* induced)



the subgraph of Γ *induced* on $\{1,2,3,4\}$

Substructures and Embeddings

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An embedding of one PLS $(\mathcal{P}, \mathcal{L}, I)$ in another, $(\widehat{\mathcal{P}}, \widehat{\mathcal{L}}, \widehat{I})$, is a pair of injections $\iota : \mathcal{P} \to \widehat{\mathcal{P}},\ \mathcal{L} \to \widehat{\mathcal{L}}$ such that $(P, \ell) \in I \Rightarrow (\iota(P), \iota(\ell)) \in \widehat{I}$. The embedding is strict if $(P, \ell) \in I \Leftrightarrow (\iota(P), \iota(\ell)) \in \widehat{I}$.

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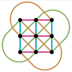
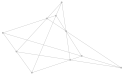
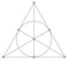
Examples of Embeddings

$\mathbb{A}^2\mathbb{F}_3 =$  embeds in $\mathbb{P}^2 F$ iff $\mathrm{char}(F) = 3$ or F has a primitive cube root of unity. (Note: $\mathbb{F}_q$ satisfies this condition iff $q \not\equiv 2 \mod 3$.)

The Desargues configuration  embeds in every finite projective plane of order $q > 2$ (strongly, for $q > 3$).

The projective plane of order two $\mathbb{P}^2\mathbb{F}_2 =$  embeds in most known finite planes. Neumann's Conjecture states that the only finite projective planes without a subplane of order two are the classical planes of odd order.

Hanna Neumann

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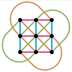
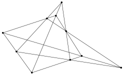
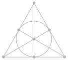
$\mathbb{A}^2\mathbb{F}_3 =$  embeds in $\mathbb{P}^2 F$ iff char$(F) = 3$ or F has a primitive cube root of unity. (Note: $\mathbb{F}_q$ satisfies this condition iff $q \not\equiv 2 \mod 3$.)

The Desargues configuration  embeds in every finite projective plane of order $q > 2$ (strongly, for $q > 3$).

The projective plane of order two $\mathbb{P}^2\mathbb{F}_2 =$  embeds in most known finite planes.
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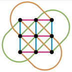
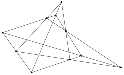
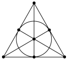
$\mathbb{A}^2\mathbb{F}_3 =$  embeds in $\mathbb{P}^2 F$ iff $\mathrm{char}(F) = 3$ or F has a primitive cube root of unity. (Note: $\mathbb{F}_q$ satisfies this condition iff $q \not\equiv 2 \mod 3$.)

The Desargues configuration  embeds in every finite projective plane of order $q > 2$ (strongly, for $q > 3$).

The projective plane of order two $\mathbb{P}^2\mathbb{F}_2 =$  embeds in most known finite planes.
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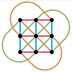
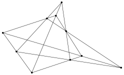
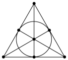
$\mathbb{A}^2\mathbb{F}_3 =$  embeds in $\mathbb{P}^2F$ iff $\mathrm{char}(F) = 3$ or F has a primitive cube root of unity. (Note: $\mathbb{F}_q$ satisfies this condition iff $q \not\equiv 2 \mod 3$.)

The Desargues configuration  embeds in every finite projective plane of order $q > 2$ (strongly, for $q > 3$).

The projective plane of order two $\mathbb{P}^2\mathbb{F}_2 =$  embeds in most known finite planes.
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Embedding Questions for a finite PLS

Every PLS embeds in a projective plane (by a process of free closure).

Open Question [Erdős 1979 and probably earlier]

Must every finite PLS embed in a *finite* projective plane?

(It is equivalent to consider only linear spaces, and to ask for a strict embedding.) Expert opinion/intuition is quite mixed regarding the answer to this question.

We know that there exist finite PLS's which do not embed in Hughes planes or André planes (a particular class of translation planes), but no finite PLS is known to not embed in any finite translation plane, even if one restricts to 'two-dimensional translation planes' (i.e. arising from line spreads of $\mathbb{P}^3\mathbb{F}_q$).

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Must every finite PLS embed in a *finite* projective plane?

(It is equivalent to consider only linear spaces, and to ask for a strict embedding.) Expert opinion/intuition is quite mixed regarding the answer to this question.

We know that there exist finite PLS's which do not embed in Hughes planes or André planes (a particular class of translation planes), but no finite PLS is known to not embed in any finite translation plane, even if one restricts to 'two-dimensional translation planes' (i.e. arising from line spreads of $\mathbb{P}^3\mathbb{F}_q$).

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Embedding Questions for a finite PLS

If one relaxes ‘spread’ to ‘partial spread’, then we are in much better shape (although the harder problem then becomes trying to extend the partial spread to a spread):

Theorem (M. and Williford, 2009)

(i) *Every finite PLS is embeddable in a finite translation net generated by a partial spread of a finite vector space.*

(ii) *Let p be prime and let $\overline{\mathbb{F}_p}$ be the algebraic closure of $\mathbb{F}_p$. Then every finite PLS is embeddable in a translation plane of finite dimension over $\overline{\mathbb{F}_p}$.*

Unfortunately in (ii), the embedding is not known to lie in a finite subplane (despite the fact that every finite subset of $\overline{\mathbb{F}_p}$ lies in a finite subfield).

Computational Complexity of Embedding Questions

Finite geometry currently suffers from a lack of any thorough investigation of the complexity of basic computational tasks, comparable to what is now available in graph theory and in much of algebra (particularly group theory)!

Given two finite planes Π and $\widetilde{\Pi}$ of order $< n$, one can answer the question 'Does Π embed in $\widetilde{\Pi}$?' in time bounded by a polynomial in n.

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Lower Bounds on Complexity of Embedding a PLS:

Given n, choose M to be a random n-bit integer, so $n = O(\log M)$. We have shown how to construct a partial linear space $\Pi(M)$ of size $O(n)$, such that in order to *construct* an embedding $\Pi(M) \hookrightarrow \mathbb{P}^2\mathbb{F}_q$ for some q, requires first factoring M. The best known algorithms for this have subexponential execution time $O(exp(cn^{1/3}\log^{2/3} n))$.

It may be possible to nonconstructively prove embeddability without producing such an embedding, but we do not know how. Our best algorithm for solving the embedding problem, requires non-polynomial (in fact subexponential) time.

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Upper Bounds on Complexity of Embedding a PLS:

Let Π be a partial linear space of size $O(n)$, and let p be prime. Then Π embeds in $\mathbb{P}^2\mathbb{F}_{p^e}$ for some $e \geqslant 1$, iff Π embeds in $\mathbb{P}^2\overline{\mathbb{F}_p}$ where $\overline{\mathbb{F}_p}$ is the algebraic closure of $\mathbb{F}_p$.

In general the best way I know how to decide existence of such an embedding, is to solve a system of polynomial equations by existing methods from computational commutative algebra. Known methods, however, are practical only for small n; both the space and time requirements are exponential ($O(exp(n^4))$ for deterministic algorithms, $O(exp(n^2))$ for nondeterministic algorithms).

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Evidence that embeddability is hard

Let Π be a finite PLS.

I believe that the question of embeddability of Π in a finite classical plane is hard to determine.

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More Open Questions

Q: Must every finite plane of order n^2 have a subplane of order n? (And a unital of order n?)

Q: Must every finite plane Π have a proper extension (i.e. does Π embed in a larger plane)?

In $\mathbb{P}^2\mathbb{F}_{p^e}$, every quadrangle generates a subplane of order p. Gleason's theorem shows that if any quadrangle in Π generates a subplane of order 2, then $\Pi \cong \mathbb{P}^2\mathbb{F}_{2^e}$.

Q: Are there any primes other than 2 for which the corresponding statement holds?

Q: If Π is a finite plane for which every quadrangle generates a *proper* subplane, must Π be classical (of order p^e, $e \geqslant 2$)?

Q: Can a plane of order $n > 3$ embed in a plane of order *not* a power of n?

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Heuristically counting subplanes of order k

Number theory abounds in hard problems for which *conjectural* answers can be deduced using certain *heuristics*. Finite geometry needs more heuristics like this:

Let k be a small positive integer. Given a plane Π, let $N_k(\Pi)$ be the number of subplanes of order k. Heuristically, if Π has order $n \gg 2$, then

$$N_2(\Pi) \approx \tfrac{1}{168} n^3(n^3-1)(n+1) \sim \tfrac{1}{168} n^7.$$

This heuristic applies best to the 'uglier' planes (those with very few automorphisms). For example:

For planes of order 25, the heuristic says $N_2(\Pi) \approx 37{,}781{,}250$. There are 193 planes of order 25 known; and ignoring the classical and Hughes planes, the number of subplanes of order 2 varies from

$$35{,}110{,}000 \quad \text{to} \quad 43{,}569{,}000.$$

The results for planes of order 49 are *much* better.

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Similar counts for larger fixed k leads to

$$N_k(\Pi) \approx c_k n^{(3-k)(k^2+k+1)}$$

as $n \to \infty$, where c_k is a positive constant depending only on k.

For example, $N_3(\Pi) = O(1)$ and $N_4(\Pi) = O(n^{-21})$.

Among the hundreds of thousands of known planes of order 49, about 1 in every 20,000 has subplanes of order 3; and no subplanes of order 4 have been found.

As $n \to \infty$, the hope of finding subplanes of order 4 seems to *decrease*, contrary to some expectations.

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My assertion is that you won't find any subplanes of order other than 2 this way. Tim doesn't believe my heuristics.

Penttila's scheme for finding planes of order $n \neq p^e$

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