Planes, Nets and Webs

Lecture 1

G. Eric Moorhouse

Department of Mathematics
University of Wyoming

Zhejiang University—March 2019

“Combinatorics is the slums of topology.”
—Henry Whitehead

Case in point: the SECC

We hope this view of combinatorics is changing thanks to the influence of people like

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Timothy Gowers

László Babai

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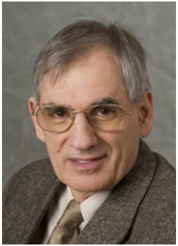

László Babai

Acknowledgements

I am grateful to those who have inspired my mathematical development:

Chat Yin Ho

William Kantor

Peter Cameron

some notation

Finite field of prime order p:	$\mathbb{F}_p$ or $\mathbb{Z}_p$ or $GF(p)$
Finite field of prime power order $q{=}p^e$:	$\mathbb{F}_q$ or $GF(q)$
Classical affine plane defined over F:	$\mathbb{A}^2F$ or $AG(2,F)$
Classical projective plane defined over F:	$\mathbb{P}^2F$ or $F\mathbb{P}^2$ or $PG(2,F)$

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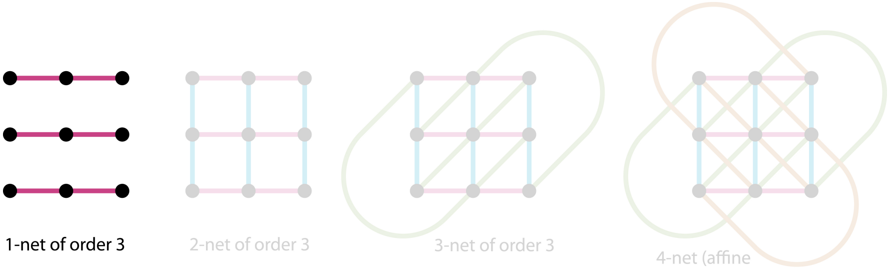

A *k*-net of order *n* has n^2 points, nk lines each with n points, in k parallel classes of n lines each. Two lines are either parallel or they meet in a unique point.

Here $k \leqslant n+1$; and an $(n+1)$-net of order n is an affine plane.

In all known cases, n is a prime power.

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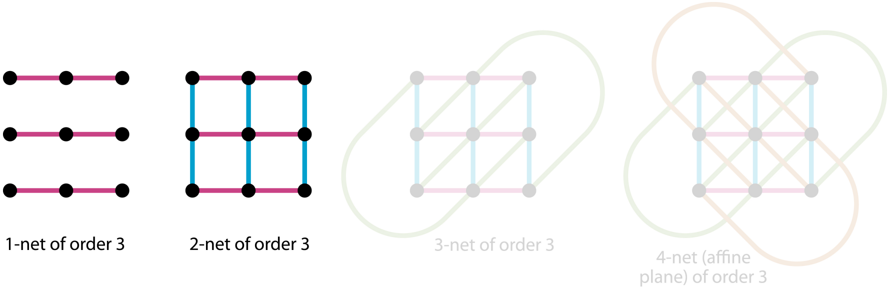

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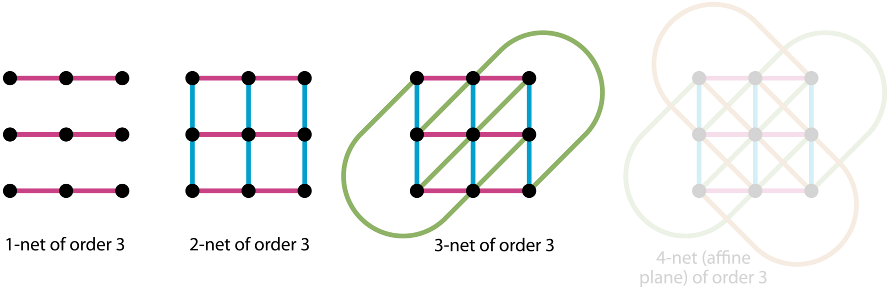

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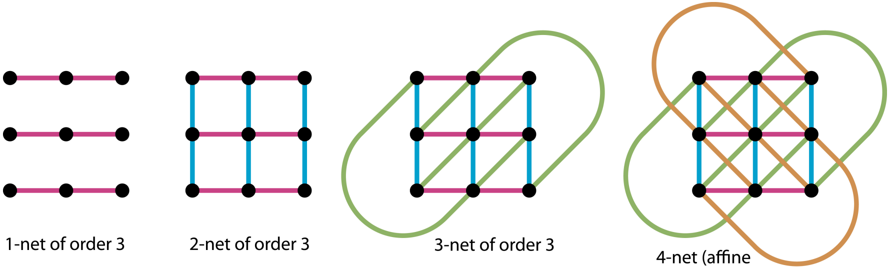

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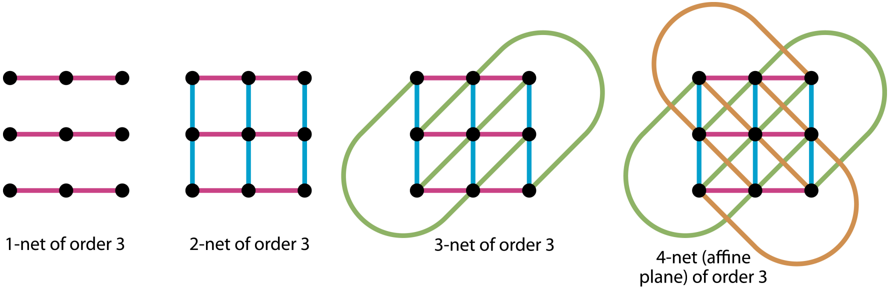

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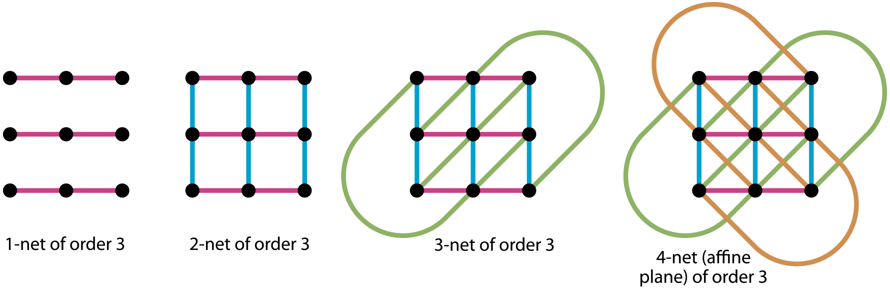

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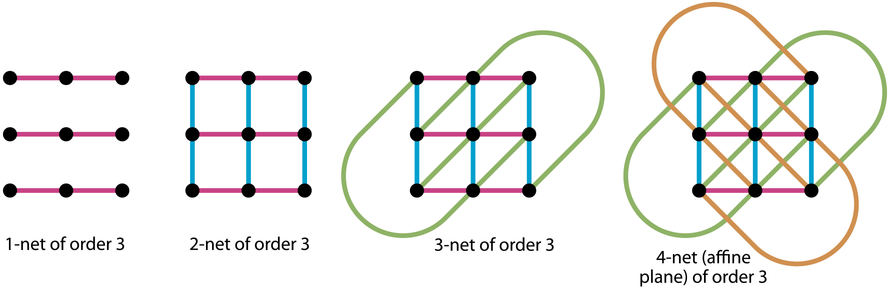

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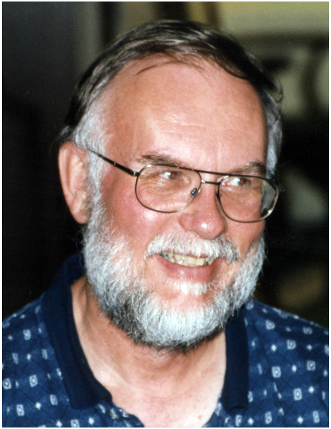
Orders of Planes



“The survival of finite geometry as an active field of study probably depends on someone finding a finite plane of non-prime-power order.”

—Gary Ebert

Orders of Planes

Clement Lam

John Thompson

Take a set of n distinct symbols, $|F| = n$. For a k-net of order n, we label points by a subset $\mathcal{N} \subseteq F^k$. Point $(a_1, a_2, \ldots, a_k) \in \mathcal{N}$ lies on line a_i of the i-th parallel class. We may assume $(0, 0, \ldots, 0) \in \mathcal{N}$.

Equivalent definition of a k-net of order n: $\mathcal{N} \subseteq F^k$, $|\mathcal{N}| = n^2$ and each vector $(a_1, a_2, \ldots, a_k) \in \mathcal{N}$ is uniquely determined by any two of its coordinates.

Unless otherwise indicated, $F = \mathbb{F}_p = \{0, 1, 2, \ldots, p-1\}$ where p is prime.

Classical affine plane $\mathbb{A}^2 F$ of order p:

$$\mathcal{N} = \{(x,\ y,\ x+y,\ 2x+y,\ \ldots,\ (p-1)x+y)\ :\ x, y \in F\}.$$

Motivating Open Question

Must every plane of prime order p be classical?

Coordinatizing Nets

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Must every plane of prime order p be classical?

(Dual) Codes of Nets

Let $\mathcal{N}$ be a k-net of prime order p. The set of all k-tuples $(f_1, f_2, \ldots, f_k)$ of functions $f_i : F \to F$ such that $f_i(0) = 0$ and

$$f_1(a_1) + f_2(a_2) + \cdots + f_k(a_k) = 0$$

for all $(a_1, a_2, \ldots, a_k) \in \mathcal{N}$ forms a vector space $\mathcal{V} = \mathcal{V}(\mathcal{N})$.

E.g. for the classical 4-net $\mathcal{N} = \{(x,\ y,\ x{+}y, x{+}\alpha y)\ :\ x, y \in F\}$ where $\alpha \neq 0, 1$, the space $\mathcal{V}$ consists of all 4-tuples (f_1, f_2, f_3, f_4) of functions $F \to F$ where

$$\begin{aligned} f_1(t) &= (a{+}b)t + (1{-}\alpha)ct^2, \\ f_2(t) &= (a{+}\alpha b)t + (\alpha{-}1)\alpha ct^2, \\ f_3(t) &= -at + \alpha ct^2, \\ f_4(t) &= -bt - ct^2. \end{aligned}$$

for some $a, b, c \in F$, so $\dim \mathcal{V} = 3$.

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Conjectured Bounds for $\dim\mathcal{V}$

Conjectured rank bound

For any k-net of prime order p,

$$\dim\mathcal{V}(\mathcal{N}_k) \leqslant \tfrac{1}{2}(k-1)(k-2).$$

Moreover for any sequence of subnets $\mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots \subset \mathcal{N}_k$,

$$\dim\mathcal{V}(\mathcal{N}_{i+1}) - \dim\mathcal{V}(\mathcal{N}_i) \leqslant i-1.$$

If this holds, then all planes of prime order are classical!

Plane curves of degree k have genus $g \leqslant \frac{1}{2}(k-1)(k-2)$. This is not a coincidence.

The analogue of the conjecture for webs (e.g. over $\mathbb{R}$ or $\mathbb{C}$) is actually a theorem. Some analogues are known in prime characteristic, e.g. for webs over $F = \mathbb{Q}_p$ or $\mathbb{F}_p(t)$.

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Codes of 3-nets

Theorem

For 3-nets of prime order p, the conjectured bound $\dim \mathcal{V}(\mathcal{N}_3) \leqslant 1$ *holds. We have equality iff the net is cyclic (i.e. a subnet of a classical plane).*

Proof. $\mathcal{N}$ is a set of p^2 triples $(x, y, z) \in F^3$, $F = \mathbb{F}_p$, such that any triple is uniquely determined by two of its coordinates. Let $(f, g, h) \in \mathcal{V}(\mathcal{N})$, so $f(0) = g(0) = h(0) = 0$ and

$$f(x) + g(y) + h(z) = 0 \quad \text{for all } (x, y, z) \in \mathcal{N}.$$

Let $\zeta = e^{2\pi i/p}$ and consider the exponential sum $S_f = \sum_{x \in F} \zeta^{f(x)}$. Then

$$S_f S_g = \sum_{x,y \in F} \zeta^{f(x)+g(y)} = p \sum_{z \in F} \zeta^{-h(z)} = p\overline{S_h}.$$

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$$S_f S_g S_h = p S_h \overline{S_h} = p|S_h|^2.$$

By symmetry, $|S_f| = |S_g| = |S_h| \in \{0, p\}$.

If $|S_f| = |S_g| = |S_h| = p$ then f, g, h are constant functions. However, $f(0) = g(0) = h(0) = 0$ so $f = g = h = 0$.

Otherwise $S_f = S_g = S_h = 0$ so f, g, h are permutations of F. We may assume $f(x) = x$, $g(y) = y$ and $h(z) = z$. Since $z = h(z) = -f(x) - g(y) = -x - y$ we get

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Progress on 4-nets of order p

Theorem

Let $\mathcal{N}_4$ be any 4-net of prime order p. Then

(i) *the number of cyclic 3-subnets is 0, 1, 3 or 4.*

(ii) *$\mathcal{N}_4$ has four cyclic 3-subnets iff $\mathcal{N}_4$ is classical (a 4-subnet of $\mathbb{A}^2\mathbb{F}_p$).*

(iii) *If $\mathcal{N}_4$ has at least one cyclic 3-subnet, then the conjectured rank bound holds.*

The rank bound is known to hold for 4-nets of small prime order p.

Much is known about the prime factorization of the associated exponential sums. And some results are known for $k > 4$.

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Phillip Griffiths


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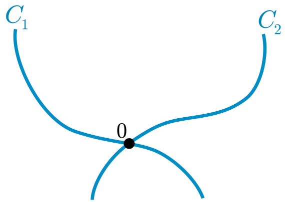
John Little

Double Translation Surfaces



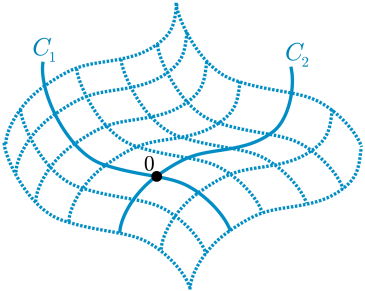
Let C_1 and C_2 be two smooth curves passing through the origin in $\mathbb{R}^d$, intersecting transversely (i.e. not having a common tangent line).

The Minkowski sum C_1+C_2 is the surface consisting of all points $u_1+u_2 \in \mathbb{R}^d$ where $u_i \in C_i$.

Suppose curves C_3 and C_4 also lie in this same surface, such that each pair of curves C_i and C_j intersects transversely at the origin.

If it happens that $C_3+C_4 = C_1+C_2$ (a very strong condition), then this surface is called a double translation surface.

Double Translation Surfaces



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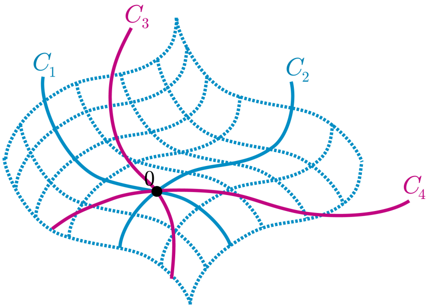

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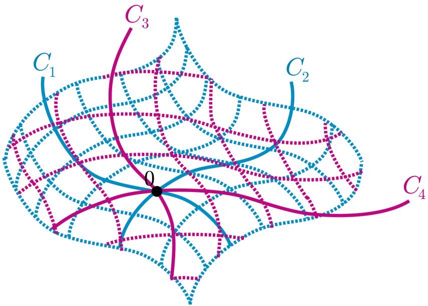

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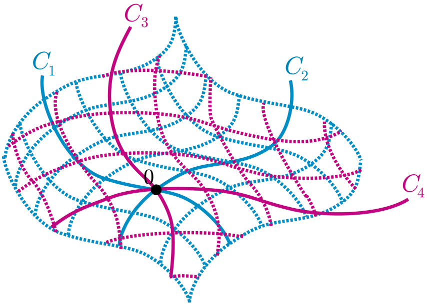

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Any double translation surface in $\mathbb{R}^d$ must lie in a subspace of dimension at most 3. When the surface spans $\mathbb{R}^3$, the tangent lines to the curves C_i meet the plane at infinity in a curve C of degree 4 and genus 3; and the surface may be recovered from C.

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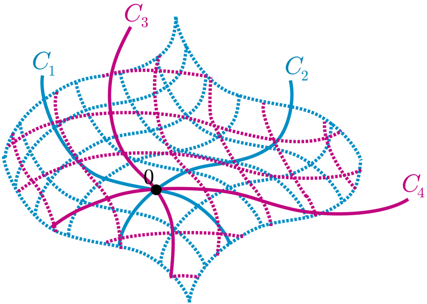

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Example 1 (Lie)

Fix $\alpha \notin \{0,1\}$. The quadric $z = \alpha x^2 - y^2$ in $\mathbb{R}^3$ is a double translation surface $C_1 + C_2 = C_3 + C_4$ where

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In this case the curve C at infinity is a singular curve of degree four with equation $XY(X-Y)(\alpha X-Y) = 0$.

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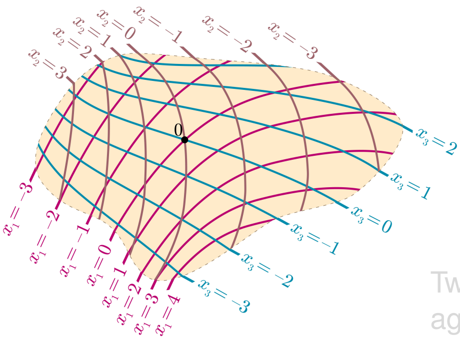
A (2-dimensional) k-web has point set $\mathcal{W} \subset \mathbb{R}^2$, an open neighbourhood of $\mathbf{0}$. It has k smooth coordinate functions $x_1, x_2, \ldots, x_k : \mathcal{W} \to \mathbb{R}$ such that for all $i \neq j$, ∇x_i and ∇x_j are linearly independent throughout $\mathcal{W}$; also $x_i(\mathbf{0}) = 0$.



A 3-web

The level curves for $x_1, x_2, \ldots, x_k$ intersect transversely, forming the 'lines' of the web.

Point $P \in \mathcal{W}$ has k coordinates $x_1(P), x_2(P), \ldots, x_k(P)$, any two of which uniquely determine the point P.

Two webs *are the same* if they agree in a neighbourhood of $\mathbf{0}$ (so only the germs of the coordinate functions x_i are relevant).

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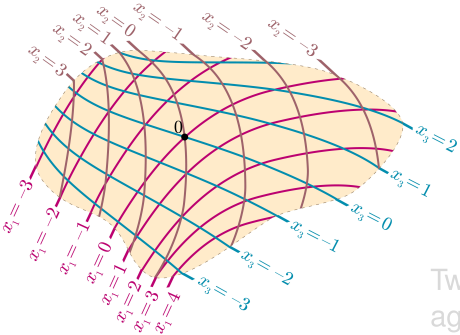

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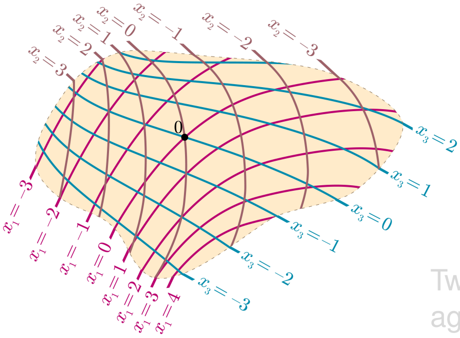

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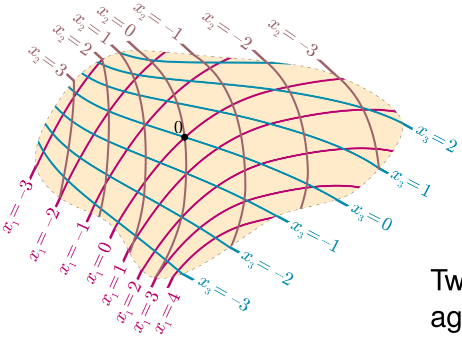

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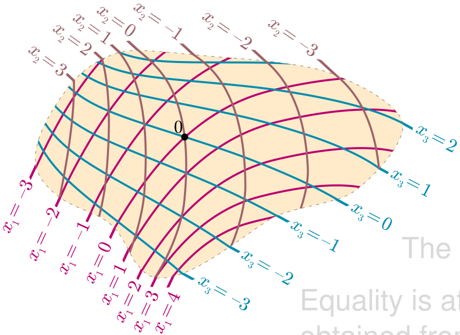

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Consider the vector space $\mathcal{V}$ consisting of all k-tuples $(f_1, f_2, \ldots, f_k)$ of smooth functions $f_i : \mathbb{R} \to \mathbb{R}$ such that $f_i(0) = 0$ and

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for all $P \in \mathcal{W}$.

The rank of $\mathcal{W}$ is $\dim \mathcal{V} \leqslant \frac{1}{2}(k-1)(k-2)$.

Equality is attained for algebraic k-webs obtained from extremal (i.e. maximal genus) plane curves of degree k. For $k \geqslant 5$, other examples are known.

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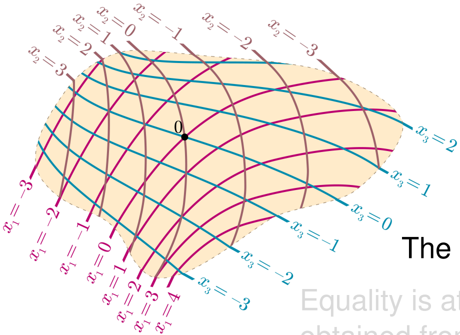

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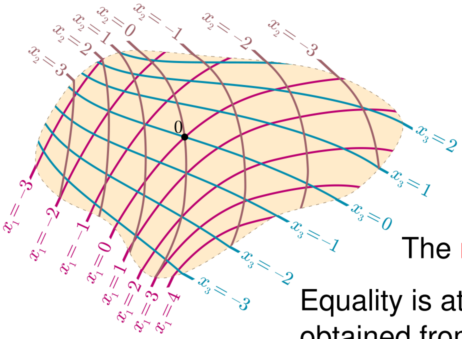

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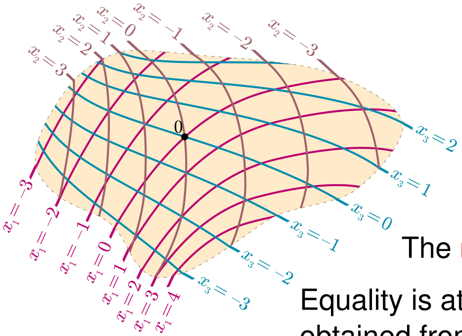

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