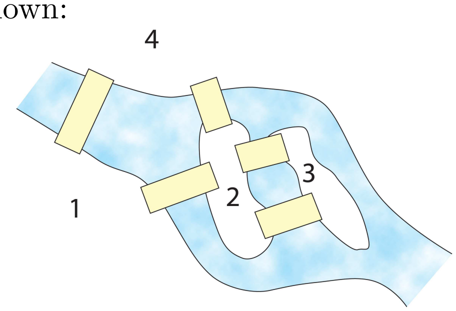

# Counting Walks (Handout April 22, 2013)

We count the number of walks of length $n$ between any two vertices in a graph.

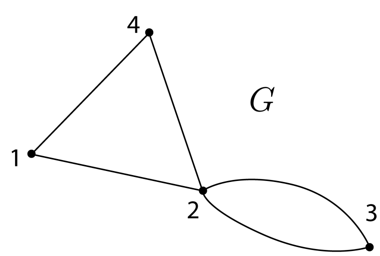
Consider the following map with four land masses (labeled 1, 2, 3 and 4) and five bridges (not labeled). For each $i, j \in [4]$, denote by $w_n(i,j)$ the number of walks of length from land mass $i$ to land mass $j$, where the *length* of a walk is the number of bridges crossed during the walk. A table of values of $w_2(i,j)$, the number of walks of length 2 from vertex $i$ to vertex $j$, is shown:




$w_2(i,j)$

| $i \backslash j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 2 | 1 | 2 | 1 |
| 2 | 1 | 6 | 0 | 1 |
| 3 | 2 | 0 | 4 | 2 |
| 4 | 1 | 1 | 2 | 2 |

We represent this map using the graph $G$ whose vertices represent land masses, and whose edges represent bridges:




so that $w_n(i,j)$ is the number of walks of length $n$ from vertex $i$ to vertex $j$ in $G$. The number of walks of length $n$ in a graph $G$ is simply expressed using matrix arithmetic, as we know explain.

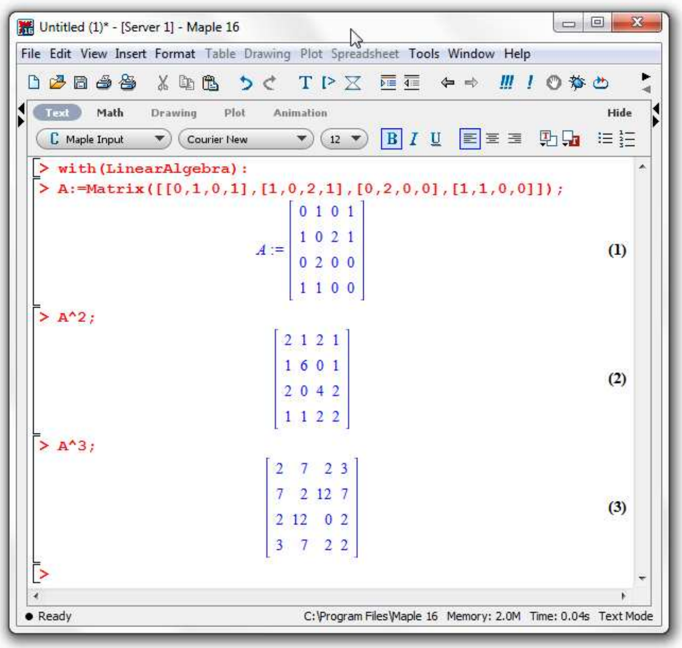
Let $G$ be a graph on $n$ vertices. We may assume that the vertices are indexed using the elements of $[n] = \{1, 2, 3, \ldots, n\}$. The *adjacency matrix* of $G$ is the $n \times n$ matrix whose $(i,j)$-entry is

$$a_{ij} = \text{number of edges from vertex } i \text{ to vertex } j.$$

In a simple graph, where no loops or multiple edges are allowed, we have $a_{ij} = 0$ or 1, and $a_{ii} = 0$; but for our present purposes, such a restriction is not needed. A *walk of length $n$ in $G$* from vertex $v$ to vertex $w$, is a sequence of $n$ edges, starting with an edge from vertex $v_0 = v$ to some vertex $v_1$, followed by an edge from $v_1$ to a vertex $v_2$, etc., and ending with an edge from vertex $v_{n-1}$ to vertex $v_n = w$. In a walk, repeated vertices and repeated edges are permitted (compare with a *trail*, where vertices may be repeated but not edges).

> **Theorem 1.** The number $w_n(i, j)$ of walks of length $n$ from vertex $i$ to vertex $j$ in a graph $G$, is the $(i, j)$-entry of $A^n$ where $A$ is the incidence matrix of $G$.

The following MAPLE code demonstrates using our graph $G$ above, where we first enter the adjacency matrix $A$ and then computes $A^2$ and $A^3$:




Note that the matrix $A^2$ yields our table of values of $w_2(i, j)$ above; and the number of walks of length 3 from vertex 1 to vertex 4, say, is $w_3(2, 4) = 7$, the $(2, 4)$-entry of $A^3$.

To see why this works, consider first the case $n = 2$. The number of walks of length 2 from vertex $i$ to vertex $j$ is

$$\begin{aligned} w_2(i,j) &= \sum_{k\in[n]} \binom{\text{number of edges}}{\text{from } i \text{ to } k} \binom{\text{number of edges}}{\text{from } k \text{ to } j} \\ &= \sum_{k\in[n]} a_{ik}a_{kj} \\ &= \text{the } (i,j)\text{-entry of } A^2 \end{aligned}$$

by the definition of matrix multiplication. Similarly for arbitrary $n$, the number of walks of length $n$ from vertex $i$ to vertex $j$ is

$$\begin{aligned} w_n(i,j) &= \sum_{v_1,v_2,\ldots,v_{n-1}\in[n]} \binom{\text{number of edges}}{\text{from } i \text{ to } v_1}\binom{\text{number of edges}}{\text{from } v_1 \text{ to } v_2}\cdots\binom{\text{number of edges}}{\text{from } v_{n-1} \text{ to } j} \\ &= \sum_{v_1,v_2,\in[n]} a_{iv_1}a_{v_1v_2}a_{v_2v_3}\cdots a_{v_{n-1}j} \\ &= \text{the } (i,j)\text{-entry of } A^n \end{aligned}$$

which proves Theorem 1. ☐

For each pair of vertices $(i,j)$, we can compute as many terms as desired of the sequence $w_0(i,j)$, $w_1(i,j)$, $w_2(i,j)$, $w_3(i,j)$, $\ldots$ by 'simply' taking successive powers of the adjacency matrix $A$, then reading off the $(i,j)$ entry. Better yet, we can explicitly obtain the (ordinary) generating function for this sequence,

$$W(x) = W_{ij}(x) = \sum_{n\geq 0} w_n(i,j)x^n = w_0(i,j) + w_1(i,j)x + w_2(i,j)x^2 + w_3(i,j)x^3 + \cdots,$$

sometimes known as the *walk generating function*.

**Theorem 2.** The generating function $W_{ij}(x)$ for $w_n(i,j)$ equals the $(i,j)$-entry of $(I - xA)^{-1}$.

*Proof.* By direct expansion we see that

$$\begin{aligned} &(I - xA)(I + xA + x^2A^2 + x^3A^3 + x^4A^4 + \cdots) \\ &\qquad = I - xA + xA - x^2A^2 + x^2A^2 - x^3A^3 - x^4A^4 + x^4A^4 - \cdots \\ &\qquad = I, \end{aligned}$$

so that

$$(I - xA)^{-1} = I + xA + x^2A^2 + x^3A^3 + x^4A^4 + \cdots.$$

The $(i,j)$-entry of this matrix is $\sum_{n\geq 0} x^n w_n(i,j) = W_{ij}(x)$. ☐

In our original example, the generating function for the number of walks of length $n$ from vertex $i$ to vertex $j$ is the $(i,j)$-entry of

$$\big(I - xA\big)^{-1} = \frac{1}{d(x)}\begin{bmatrix} 1-5x^2 & x(1+x) & 2x^2(1+x) & x(1+x-4x^2) \\ x(1+x) & 1-x^2 & 2x(1-x^2) & x(1+x) \\ 2x^2(1+x) & 2x(1-x^2) & (1+x^2)(1-2x) & 2x^2(1+x) \\ x(1+x-4x^2) & x(1+x) & 2x^2(1+x) & 1-5x^2 \end{bmatrix}$$

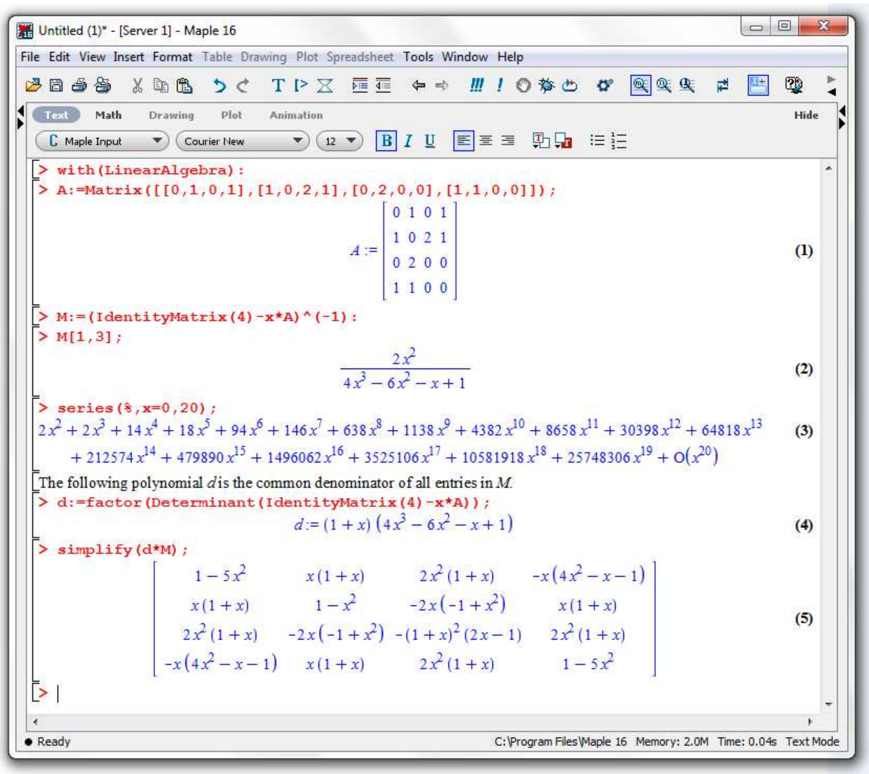
where the common denominator $d(x) = (1+x)(1-x-6x^2+4x^3)$. In particular

$$W_{13}(x) = 2x^2 + 2x^3 + 14x^4 + 18x^5 + 94x^6 + 146x^7 + 638x^8 + 1138x^9 + 4382x^{10} + \cdots,$$

so the number of walks of length $n$ from vertex 1 to vertex 3 is given by

$$0, 0, 2, 2, 14, 18, 94, 146, 638, 1138, 4382, \ldots$$

for $n = 0, 1, 2, 3, \ldots$. All these computations are demonstrated in the MAPLE session




This concludes the solution of the 4-part problem Will solved in the 1997 film *Good Will Hunting*.