The Tychonoff Plank

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G. Eric Moorhouse, University of Wyoming

Begin with an ordinal

$$\gamma = \{x : x \text{ is an ordinal and } x < \gamma\}.$$

Every ordinal is a canonical representative of the order type of some well-ordered set; and what is important here is that γ is a well-ordered set with least element 0. Since each ordinal is the set of all smaller ordinals, we have

$$\gamma = [0,\gamma) \quad \text{and} \quad \gamma+1 = \gamma \cup \{\gamma\} = [0,\gamma].$$

We use the standard interval notation for subsets of γ:

$$(a,b) = \{x \in \gamma \ : \ a < x < b\}; \quad [a,b] = \{x \in \gamma \ : \ a \leqslant x \leqslant b\}$$

etc. The **order topology** on γ has as basis the intervals of the form (a,b), $[0,b)$ contained in γ. (I haven't bothered to write $(a,b]$ since this is the same as $(a,b{+}1)$. Also note that $(a,b) = [a{+}1,b)$.) For example

$$\omega = \{0,1,2,\ldots\} = \{\text{non-negative integers}\}$$

and

$$\omega{+}1 = [0,\omega] = \omega \cup \{\omega\}$$

which is the one-point compactification of ω. The proof of the following is very similar to our proof that $X = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$ is compact (even though X is not well-ordered).

Theorem 1. For every ordinal γ, the space $[0,\gamma]$ is compact.

Proof. Let $\mathcal{O}$ be an open cover of $[0,\gamma]$, and set

$$S = \{a \in [0,\gamma] \ : \ [a,\gamma] \text{ is covered by finitely many sets in } \mathcal{O}\}.$$

Since the point $[\gamma,\gamma] = \{\gamma\}$ is covered by some member of $\mathcal{O}$, we have $\gamma \in S$ and in particular $S \neq \varnothing$. Since $[0,\gamma]$ is well-ordered, S has a least element which we denote by m. If $m = 0$ then we are done, so assume otherwise. Since $m \in S$, we have

$$[m,\gamma] \subseteq U_1 \cup U_2 \cup \cdots \cup U_n$$

for some $U_1, U_2, \ldots, U_n \in \mathcal{O}$. If m is a successor ordinal, say $m = a+1$ where $a \in [0, \gamma]$, then $a \in U$ for some $U \in \mathcal{O}$, and then

$$[a, \gamma] = \{a\} \cup [m, \gamma] \subseteq U \cup U_1 \cup U_2 \cup \cdots \cup U_n$$

so $a \in S$, contradicting the minimality of $m \in S$. On the other hand if m is a limit ordinal, then $m \in U$ for some $U \in \mathcal{O}$ and U contains a basic open neighbourhood of m of the form (a, b) with $a < m < b$. In this case we may choose $r \in (a, m)$ (since m is a limit ordinal, there are infinitely many such r's to choose from) and

$$[r+1, \gamma] = (r, m) \cup [m, \gamma] \subseteq U \cup U_1 \cup U_2 \cup \cdots \cup U_n.$$

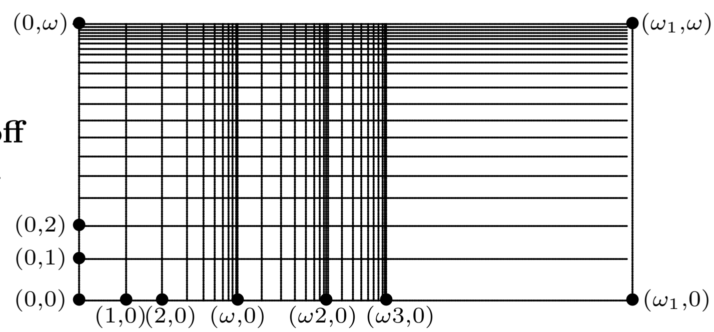
Once again, $r+1 \in S$ contradicts the minimality of m. The result follows. □

The smallest uncountable ordinal is denoted ω_1. The Tychonoff Plank is the topological space

$$X = [0, \omega_1] \times [0, \omega].$$

Note that X has the product topology. In particular, recall that each of the 'horizontal lines' $[0, \omega_1] \times \{\beta\}$ is homeomorphic to $[0, \omega_1]$; and each of the 'vertical lines' $\{\alpha\} \times [0, \omega]$ is homeomorphic to $[0, \omega]$.

The Tychonoff Plank



Since both $[0, \omega_1]$ and $[0, \omega]$ are compact Hausdorff, so is X. It follows that X is normal, and hence completely regular by Urysohn's Lemma.

Not every subspace of a normal space is normal. The standard counterexample for demonstrating this is the Punctured Tychonoff Plank

$$Y = X \smallsetminus \{(\omega_1, \omega)\}.$$

To show that Y is not normal, consider the subsets

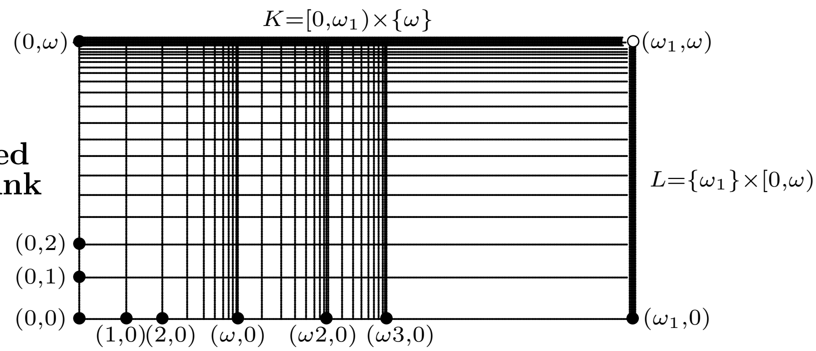
$$K = [0, \omega_1) \times \{\omega\}, \quad L = \{\omega_1\} \times [0, \omega).$$

These sets are closed in Y since they are obtained by intersecting Y with the closed sets

$$[0,\omega_1]\times\{\omega\},\ \{\omega_1\}\times[0,\omega]\subset X$$

respectively.

The Punctured Tychonoff Plank



Suppose that $K\subseteq U$ and $L\subseteq V$ for some disjoint open sets $U,V\subset Y$. For each $n\in\omega$ there exists $\alpha_n<\omega_1$ such that

$$[\alpha_n,\omega_1]\times\{n\}\subseteq V.$$

Now let

$$\beta=\sup\{\alpha_n\ :\ n\in\omega\}=\bigcup\{\alpha_n\ :\ n\in\omega\}.$$

Since the ordinals are totally ordered by inclusion, β is an ordinal; and since β is a countable union of countable sets (recall that each $\alpha_n<\omega_1$), β is countable, whence $\beta<\omega_1$. By definition of β, we have $(\beta,n)\in V$ for every $n\in\omega$. But the sequence of points $\big((\beta,n)\big)_{n\in\omega}$ converges to $(\beta,\omega)\in K$, so there exists $n\in\omega$ such that $(\beta,n)\in U$. Now $U\cap V$ contains a point (β,n), a contradiction.

Since $Y\subset X$ is not normal we see that

- subspaces of normal spaces are not necessarily normal; and
- completely regular spaces are not necessarily normal.