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Tychonoff's Theorem: The General Case

The following proof of Tychonoff's Theorem uses transfinite induction. Let $X = \prod_{i\in\alpha} X_i$ where each X_i is compact. (We assume each X_i is nonempty; otherwise X is also empty and the result follows trivially.) Informally we view each point $x \in X$ as an infinite tuple $x = (x_i)_{i\in\alpha}$ where $x_i \in X_i$. (Formally, x is a function whose value at $i \in \alpha$ is an element $x_i \in X_i$.) The index set α may be taken to be well-ordered. (The ordering of α has no effect on the structure of X as a topological space; its only purpose is to allow us to use induction.) And because the canonical choices of well-ordered sets are ordinals, we may in fact take α to be an ordinal; thus $X = \prod_{i\in\alpha} X_i = \prod_{i<\alpha} X_i$. (Recall that for ordinals α and β, the definition of the order relation may be written as $\beta < \alpha$ iff $\beta \in \alpha$.)

The recursion at step β depends on the type of the ordinal β. Every ordinal has exactly one of three types, as follows:

(i) *zero*, i.e. $0 = \varnothing$;

(ii) a *successor ordinal*, i.e. an ordinal of the form $\gamma + 1 = \gamma \cup \{\gamma\}$ for some ordinal γ. Examples include $1, 2, 3, \ldots; \omega + 1, \omega + 2, \ldots$; or

(iii) a *limit ordinal* (an infinite ordinal which is the union of all smaller ordinals). Examples include $\omega, 2\omega, \ldots$.

If an ordinal is not a successor ordinal, then it is the union of all smaller ordinals; this property characterizes both case (i) (since we can view zero as the union of all smaller ordinals, i.e. an empty union) and case (iii).

We also make use of the following version of the

Tube Lemma. *Let Y and Z be topological spaces, with Y compact. Suppose that $\mathcal{A}$ is a cover of $Y \times Z$ by basic open sets of the form $U \times V$, where $U \subseteq Y$ is open and $V \subseteq Z$ is open. Suppose further that no finite subcollection of $\mathcal{A}$ covers $Y \times Z$. Then there exists $y \in Y$ such that no finite subcollection of $\mathcal{A}$ covers the subspace $\{y\} \times Z$.*

Proof. Suppose that every such 'cross section' $\{y\}\times Z$ is covered by finitely many members of $\mathcal{A}$. If $\{y\} \times Z$ is covered by the basic open sets $U_i \times V_i \in \mathcal{A}$ for $i = 1, 2, \ldots, n$, then these sets actually cover a 'tube' of the form $U_y \times Z$ where $U_y = U_1 \cap U_2 \cap \cdots \cap U_n$ is an open neighbourhood of y in Y. Do this for each such cross section $\{y\} \times Z$. Since Y is compact, we have

$$Y = U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_m}$$

for some $y_1, y_2, \ldots, y_m \in Y$. Now each of the tubes $U_{y_i} \times Z$ is covered by finitely many members of $\mathcal{A}$, so $Y \times Z$ is also covered by finitely many members of $\mathcal{A}$, a contradiction. $\square$

Theorem (Tychonoff). *Assuming each X_i is compact, the product space $X = \prod_{i<\alpha} X_i$ is compact.*

Proof. Suppose, on the contrary, that X is not compact. Then there exists a cover $\mathcal{A}$ of X consisting of basic open sets, but having no finite subcover. We will obtain a contradiction.

Each point $p = (p_i)_{i\in\alpha} \in X$ gives rise to a decreasing chain of subspaces

$$Z_\beta = \Big(\prod_{i<\beta}\{p_i\}\Big) \times \Big(\prod_{\beta\leqslant i<\alpha} X_i\Big),$$

for all $\beta \leqslant \alpha$; this means that $Z_\beta \supseteq Z_\gamma$ whenever $\beta \leqslant \gamma \leqslant \alpha$. Note also that $Z_0 = X$ and $Z_\alpha = \{p\}$. We will construct a point p by choosing the coordinates p_i recursively, in such a way that none of the subspaces Z_β is covered by any finite subcollection of $\mathcal{A}$. Taking $\beta = \alpha$ gives the desired contradiction, since it only takes one member of $\mathcal{A}$ to cover the singleton $Z_\alpha = \{p\}$.

At stage β of the recursion, we specify Z_β by choosing all coordinates p_i for $i < \beta$.

Case (i): $\beta = 0$

In this case there is nothing to do since there are no indices $i < 0$. Note that by hypothesis, no finite subcollection of $\mathcal{A}$ covers $Z_0 = X$.

Case (ii): Successor Ordinal $\beta = \gamma + 1 \leqslant \alpha$

We assume that p_i has already been chosen for $i < \gamma$ in such a way that the subspace

$$Z_\gamma = \Big(\prod_{i<\gamma}\{p_i\}\Big) \times \Big(\prod_{\gamma\leqslant i<\alpha} X_i\Big)$$

is not covered by any finite subcollection of $\mathcal{A}$. Next we must choose $p_\gamma \in X_\gamma$ such that the cross section $Z_{\gamma+1} = \Big(\prod_{i\leqslant\gamma}\{p_i\}\Big) \times \Big(\prod_{\gamma<i<\alpha} X_i\Big)$ is not covered by any finite subcollection of $\mathcal{A}$. This follows by the Tube Lemma, applied in the case $Y = \big(\prod_{i<\gamma}\{p_i\}\big) \times X_\gamma \simeq X_\gamma$ and $Z = \prod_{\gamma<i<\alpha} X_i$.

Case (iii): Limit Ordinal $\beta \leqslant \alpha$

In this case the coordinates p_γ have already been chosen for $\gamma < \beta$, in such a way that none of the resulting cross sections Z_γ can be covered by finitely many members of $\mathcal{A}$. We must however verify that no finite subcollection of $\mathcal{A}$ covers the subspace

$$Z_\beta = \Big(\prod_{i<\beta}\{p_i\}\Big) \times \Big(\prod_{\beta\leqslant i<\alpha} X_i\Big).$$

Suppose, on the contrary, that Z_β is covered by $A_1, A_2, \ldots, A_n \in \mathcal{A}$. Each A_j has the form

$$A_j = \left(\prod_{s \in S_j} U_s \right) \times \left(\prod_{s \notin S_j} X_s \right)$$

for some finite subset $S_j \subseteq \alpha$. Here $U_s \subseteq X_s$ is open. Each S_j has only finitely many elements less than β; so let γ_j be the largest element of A_j less than β. Write $\gamma = \max\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, so that $\gamma + 1 < \beta$. We will show that $A_1 \cup A_2 \cup \cdots \cup A_n \supseteq Z_{\gamma+1}$ a contradiction as desired. Let $z \in Z_{\gamma+1}$, so that $z_i = p_i$ for all $i \leqslant \gamma$. There exists $\widehat{z} \in Z_\beta$ such that

$$\widehat{z}_i = \begin{cases} p_i = z_i\,, & \text{for all } i \leqslant \gamma; \\ p_i\,, & \text{whenever } \gamma < i < \beta; \text{ and} \\ z_i\,, & \text{for all } i \geqslant \beta. \end{cases}$$

Since $Z_\beta = A_1 \cup A_2 \cup \cdots \cup A_n$, we have $\widehat{z} \in A_j$ for some j. But then $z \in A_j$ since in the definition of A_j there is no restriction on the i-th coordinate for $\gamma < i < \beta$. We have obtained the desired contradiction $A_1 \cup A_2 \cup \cdots \cup A_n \supseteq Z_{\gamma+1}$.

In Case (iii) we conclude that, as claimed, no finite subcollection of $\mathcal{A}$ covers Z_β. The proof follows. $\square$

The main proof of Tychonoff's Theorem presented in Munkres' book, pages 230-235, uses Zorn's Lemma instead of transfinite induction. This alternative proof is suggested by an exercise on pages 236-237. I have simplified Munkres' notation somewhat. [Note that his Y_β is the same thing as $Z_{\beta+1}$, and it seems silly to have two ways of writing the same thing. Moreover, Munkres has insisted that the index set α have a largest element; this is not necessary for our argument, and it doesn't really shorten the proof. All it does is guarantee that the last step of the recursion lies in Case (ii), rather than Case (iii); but this is not important to the proof. Note that since we are taking α to be an ordinal, to say that α has a largest element means that it is a successor ordinal.]