Transfinite Induction

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Notes for ACNT Seminar, 20 Jan 2009

Abstract

Let $X \subset \mathbb{R}^3$ be the complement of a single point. We prove, by transfinite induction, that X can be partitioned into lines. This result is intended as an introduction to transfinite induction.

1 Cardinality

Let A and B be sets. We write

$|A| = |B|$ (in words, *A and B have the same cardinality*) if there exists a bijection $A \to B$;

$|A| \leqslant |B|$ if there exists an injection (a one-to-one map) $A \to B$;

$|A| < |B|$ if $|A| \leqslant |B|$ but there is no bijection from A to B.

The Cantor-Bernstein-Schroeder Theorem asserts that if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$. (The proof is elementary but requires some thought.) So we may reasonably speak of the cardinality of a set, as a measure of its size, and compare any two sets according to their cardinalities. The cardinality of $\mathbb{Z}$ is denoted $\aleph_0$. Every set A having $|A| \leqslant \aleph_0$ is called *countable*; in this case either A is finite, or $|A| = \aleph_0$ (*countably infinite*). Examples of countably infinite sets include {positive integers}, $2\mathbb{Z}$ = {even integers}, and $\mathbb{Q}$. Cantor showed that $\mathbb{R}$ is uncountable; we denote $|\mathbb{R}| = 2^{\aleph_0}$. This cardinality, which we call the *cardinality of the continuum*, strictly exceeds $\aleph_0$; it is also the cardinality of $\mathbb{R}^n$ for every positive integer n.

In this context it is almost obligatory to mention the *Continuum Hypothesis* (CH), which is the assertion that there is no set A satisfying $\aleph_0 < |A| < 2^{\aleph_0}$.

The validity of this statement is independent of ZFC[1], but not to worry! since we have no immediate use for CH.

2 Well-Ordering

Let S be a set with binary relation '$\leqslant$'. We say that S is *well-ordered* by the relation '$\leqslant$' if

(WO1) For all $x, y \in S$, we have $x \leqslant y$ or $y \leqslant x$.

(WO2) We have $x \leqslant y$ and $y \leqslant x$, iff $x = y$.

(WO3) For all $x, y, z \in S$ such that $x \leqslant y$ and $y \leqslant z$, we must have $x \leqslant z$.

(WO4) Every nonempty subset of S has a least element. That is, if $\varnothing \neq A \subseteq S$, then there exists $m \in A$ such that $m \leqslant x$ for all $x \in A$.

The set $\mathbb{R}$ is not well-ordered, with the usual '$\leqslant$' relation; nor are either of the subsets $\mathbb{Z}$ or $(0, \infty)$, as none of these has a least element. However, several subsets of $\mathbb{R}$ are well-ordered, most notably the set of non-negative integers[2]

$$\omega = \{0, 1, 2, \ldots\}.$$

It is the well-ordering of ω that allows us to prove statements over the non-negative integers by induction. More examples of well-ordered sets arise as subsets of ω. (More generally, every subset of a well-ordered set is well-ordered.)

We have used the *non-strict* order relation '$\leqslant$' in our definition. Sometimes it is more convenient to define a well-ordering in terms of the *strict* order relation '$<$'. Note that $x \leqslant y$ is equivalent to '$x < y$ or $x = y$'.

Another subset of $\mathbb{R}$, well-ordered by the natural relation '$\leqslant$', is $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$, pictured here (not quite to scale):

[1] ZF is the common axiomatic foundation for everyday mathematics. Adding to this the Axiom of Choice (see Section 3) gives ZFC. The work of Kurt Gödel (1940) and Paul Cohen (1963) shows that CH cannot be either proved or disproved within ZFC (unless, of course, ZFC is inconsistent, which seems unlikely).

[2] The name 'ω' for this familiar set, comes from the theory of ordinals; see Section 4.

The hollow dot indicates the omission of the limit point 1 from this set. But if we include 1, we obtain the set $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \cup \{1\}$, which is also well-ordered:

Extending further, we obtain the example $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \cup \{1, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, 1\frac{4}{5}, \ldots\}$ which is also well-ordered:

Our example $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$ above, has the same *order type* as ω, meaning that there is an order-preserving bijection $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \to \omega$ (an *order isomorphism*). The next two examples are order-isomorphic to the ordinals $\omega + 1$ and $\omega 2$, respectively (Section 4). In fact, every well-ordered set is canonically represented by a unique ordinal.

3 Transfinite Induction

The *Well-Ordering Principle* asserts that every set S can be well-ordered; that is, if S is any set, then there exists a well-ordered set A which serves as an index set for the elements of S, so we may write

$$S = \{s_\alpha : \alpha \in A\}.$$

This principle is logically equivalent to the Axiom of Choice (C), or Zorn's Lemma. Briefly, C is the assertion that for any collection S of sets, we may construct a new set S whose elements are representatives of the sets $S \in S$. If this sounds reasonable, then you probably won't need to worry any more about C for our discussion. We won't describe Zorn's Lemma here.

Our immediate interest is in the Principle of Transfinite Induction, and how it is made possible by well-ordering. Let $\{P_\alpha : \alpha \in A\}$ be a well-ordered collection of statements (so A is a well-ordered set indexing the family of statements P_α). The *Principle of Transfinite Induction* asserts that if $\bigwedge_{\alpha<\beta} P_\beta$ implies P_β, for all $\beta \in A$, then in fact P_α holds for all $\alpha \in A$. Here $\bigwedge_{\alpha<\beta} P_\beta$ is the conjunction of the statements P_α for all $\alpha < \beta$, $\alpha \in A$ (i.e. the assertion that P_α holds for all $\alpha < \beta$). This Principle probably seems quite reasonable. To justify it using well-ordering, suppose that on the contrary, P_α fails for some $\alpha \in A$. Then the set S consisting of all $\alpha \in A$ for which P_α is false, satisfies $S \neq \emptyset$ by assumption. Then S has a least element $m \in S$. Now P_β holds for all $\beta < m$, whereas P_m fails, a contradiction.

Take a moment to convince yourself that the usual induction over the non-negative integers arises a special case of this Principle, in the case $A = \omega$. In this case our hypothesis is that

$$\begin{aligned} P_0 &\quad \text{holds;} \\ P_0 &\quad \text{implies } P_1; \\ P_0 \wedge P_1 &\quad \text{implies } P_2; \\ P_0 \wedge P_1 \wedge P_2 &\quad \text{implies } P_3; \end{aligned}$$

etc. From this we are to conclude that P_α holds for every $\alpha \in \omega$; and this should be recognized as the usual principle of induction (or what is sometimes called 'complete induction').

4 Ordinals

An *ordinal* may be defined as a set S such that $x \subseteq S$ whenever $x \in S$, and the elements of S are well-ordered by the relation '$\in$'. This is one of many equivalent definitions. It is probably easier to define ordinals recursively (following von Neumann) by saying that each ordinal is the well-ordered set of all smaller ordinals. The smallest ordinal is

$$0 = \emptyset.$$

The next-smallest ordinal is

$$1 = \{0\} = \{\emptyset\},$$

followed by

$$\begin{aligned} &2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}, \\ &3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \\ &4 = \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}, \end{aligned}$$

etc. After all finite ordinals have been constructed, we continue with

$$\begin{aligned} \omega &= \{0, 1, 2, 3, \ldots\}, \\ \omega{+}1 &= \{0, 1, 2, 3, \ldots\} \cup \{\omega\}, \\ \omega{+}2 &= \{0, 1, 2, 3, \ldots\} \cup \{\omega, \omega{+}1\}, \end{aligned}$$

and eventually

$$\omega 2 = \{0, 1, 2, 3, \ldots\} \cup \{\omega,\ \omega{+}1,\ \omega{+}2,\ \omega{+}3,\ \ldots\}.$$

This is followed by $\omega 2{+}1$, $\omega 2{+}2$, $\ldots$; $\omega 3, \ldots$; $\omega 4, \ldots$; and eventually ω^2. Much later we come to ω^3, $\ldots$, ω^4, $\ldots$, and eventually ω^ω. Much later, we reach ω^{ω^ω}. And so on. All the ordinals we have mentioned so far are, however, still only countably infinite; so we have still only scratched the surface of the list of ordinals. After all countable ordinals have been defined, we meet the first uncountable ordinal, denoted ω_1. (Actually ω is an abbreviation for ω_0.) Much later we reach ω_2, the first ordinal whose cardinality exceeds that of ω_1; and so on. After $\omega_0, \omega_1, \omega_2, \ldots$ we find ω_ω. And so on...

Let β be an ordinal. Then the *successor* of β is $\beta + 1 = \beta \cup \{\beta\}$, this being the smallest ordinal exceeding β. Every ordinal is either a successor ordinal or a limit ordinal, but never both. A *limit ordinal* is an ordinal α such that $\alpha = \bigcup_{\beta<\alpha} \beta$. Note that 0 and ω are limit ordinals. Examples of successor ordinals are 1, 2, 3, etc; also $\omega{+}1$, $\omega{+}2$, etc.

Now an arbitrary set S may be indexed as

$$S = \{s_\alpha : \alpha \in A\}$$

where A is an ordinal. Moreover we may assume A is minimal among all ordinals of cardinality $|A|$; otherwise we may simply re-index suitably. (Note that the ordinals $\alpha \in A$ such that $|\alpha| \leqslant A + 1$, gives a non-empty subset of A; so by well-ordering, it has a least element. This least element can replace A as an index set for S.)

5 Example

Theorem 5.1 *Let $X = \mathbb{R}^3 \setminus \{O\}$, the complement of a point $O \in \mathbb{R}^3$. Then X can be partitioned into Euclidean lines.*

We will prove Theorem 5.1 by transfinite induction. (If you have a constructive proof of this, I would very much like to see it!) For the inductive step in our proof, will make use of the following.

Lemma 5.1 *Let $X = \mathbb{R}^3 \setminus \{O\}$, the complement of a point $O \in \mathbb{R}^3$. Let Σ be a set of mutually disjoint Euclidean lines in X, such that $|\Sigma| < 2^{\aleph_0}$. Then there exists a line $\ell \subset X$ not meeting any line of Σ.*

Proof. We first choose a point $P \in \mathbb{R}^3$, distinct from O, and not lying on any line of Σ. (For example, let $S \in \mathbb{R}^3$ be a 2-sphere. Note that $|S| = 2^{\aleph_0}$, and each line of Σ meets S in at most 2 points. So the lines of Σ meet S in at most $2|S| < 2^{\aleph_0}$ points. We may choose $P \in S$ distinct from these points of intersection, and also distinct from O.) Now consider a cone $C \subset \mathbb{R}^3$ with vertex P. This cone has $2^{\aleph_0}$ 'ruling lines', all of which pass through P. Each line of Σ meets C in at most 2 points, so clearly we may choose a ruling line ℓ of C not passing through O, and not meeting any line of Σ. □

We are now ready to prove Theorem 5.1. First we well-order the points of $X = \mathbb{R}^3 \setminus \{O\}$ as

$$X = \{P_\alpha : \alpha \in A\}.$$

We may further assume that A is the smallest ordinal of cardinality $2^{\aleph_0}$. We will recursively define sets Σ_α (for $\alpha \in A$) consisting of mutually disjoint lines in X, such that

(i) $|\Sigma_\alpha| \leqslant |\alpha| < 2^{\aleph_0}$;

(ii) P_α lies on some line of Σ_α, for all $\alpha \in A$; and

(iii) $\Sigma_\alpha \subseteq \Sigma_\beta$, whenever $\alpha < \beta,\ \alpha, \beta \in A$.

To do this, let $\alpha \in A$; we want to define Σ_α. If α is a limit ordinal, we set $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_\beta$.

Now consider a successor ordinal $\alpha = \beta + 1$. If $P_\beta \in \bigcup \Sigma_\beta$, we simply take $\Sigma_{\beta+1} = \Sigma_\beta$. Otherwise by Lemma 5.1 we may choose a line ℓ in X disjoint from $\{O\} \cup (\bigcup \Sigma_\beta)$, and we take $\Sigma_{\beta+1} = \Sigma_\beta \cup \{\ell\}$. The required properties (i), (ii), (iii) hold by induction.

Clearly the set of lines $\Sigma = \bigcup_{\alpha\in A} \Sigma_\alpha$ is a partition of X into lines.