Transcendence of e and π

Despite the fact that most real numbers (and most complex numbers) are transcendental, it is typically very hard to verify the transcendence of any specific number. The first specific number shown (by Liouville in 1844) to be transcendental, is the constant

$$\sum_{n=1}^{\infty} 10^{-n!} = 0.\underset{\hat{1}}{1}\underset{\hat{2}}{1}000\underset{\hat{6}}{1}00000000000000000\underset{\hat{24}}{1}000\ldots;$$

but this constant is of essentially no value other than what it was contrived for: to easily verify its transcendence. The most important transcendental numbers in nature are e and π, which were shown to be transcendental by Hermite and Lindemann in 1873 and 1882, respectively. It is now known that e^π is transcendental, but it is not known whether or not π^e is transcendental; for all we know, π^e might even be rational! (but probably not). Similarly, we do not know that $\pi + e$ is irrational, or that πe is irrational; but we *do* that at least one of $\pi + e$ and πe is transcendental!


Charles Hermite (1822–1901)


Ferdinand von Lindemann (1852–1939)

The proofs of transcendence of e and of π that we give here, are simplified versions[1] of the original proofs due to Hermite and Lindemann. Before tackling the more difficult questions of transcendence, we warm up by proving irrationality. And since e is easier than π, we start with:

Theorem. The number e is irrational.

[1] See I. Stewart, *Galois Theory,* 3rd ed., Chapman & Hall, 2004. The proof of the transcendence of π given by Stewart, is adapted from I. Niven, 'The transcendence of π', *Amer. Math. Monthly* **46** (1939), 469–471. To Stewart's exposition I have added some corrections and further explanations, and probably my own mistakes.

Proof. Suppose $e = \frac{a}{b}$ where a, b are relatively prime positive integers. We must have $b \geqslant 2$ since $e \notin \mathbb{Z}$. Multiplying $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ by $b!$, we obtain

$$\underbrace{(b-1)!a}_{\text{integer}} = b!e = \underbrace{b! + b! + \frac{b!}{2!} + \frac{b!}{3!} + \cdots + b + 1}_{\text{integer}} + \underbrace{\frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots}_{\text{fractional terms}}.$$

This forces the fractional terms on the right to sum to an integer; however,

$$\begin{aligned} 0 &< \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \\ &< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots \quad \text{(a geometric series)} \\ &= \frac{1}{b} \\ &< 1, \end{aligned}$$

contradicting our deduction that this sum is an integer. □

We will make frequent use of *Leibniz' Formula* for the k-th derivative of a product:

$$\frac{d^k}{dx^k} u(x)v(x) = \sum_{j=0}^{k} \binom{k}{j} u^{(j)}(x) v^{(k-j)}(x).$$

This is easily proved by induction, using the usual product rule for differentiation.

Lemma. Let $\frac{a}{b} \in \mathbb{Q}$ be a reduced fraction, and $n \geqslant 0$. Define $f(x) = \frac{1}{n!} x^n (a - bx)^n$. Then for every $k \geqslant 0$, the k-th derivative $f^{(k)}$ satisfies $f^{(k)}(0) = (-1)^k f^{(k)}\big(\frac{a}{b}\big) \in \mathbb{Z}$.

Proof. Observe that

$$f\big(\tfrac{a}{b} - x\big) = \tfrac{1}{n!}\big(\tfrac{a}{b} - x\big)^n (a - a + bx)^n = \tfrac{1}{n!}\big(a - bx\big)^n x^n = f(x).$$

Taking the k-th derivative yields

$$(*) \qquad f^{(k)}\big(\tfrac{a}{b}\big) = (-1)^k f^{(k)}(0).$$

Now write $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{n!} x^n$ and $v(x) = (a - bx)^n$. Since

$$u^{(j)}(0) = \begin{cases} 1, & \text{if } j = n; \\ 0, & \text{otherwise,} \end{cases}$$

by Leibniz' Formula we obtain

$$f^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} u^{(j)}(0) v^{(k-j)}(0) = \binom{k}{n} v^{(k-n)}(0).$$

Since $\deg v(x) = n$, we have $f^{(k)}(0) = 0$ unless $k \leqslant n \leqslant 2n$, in which case

$$f^{(k)}(0) = \binom{k}{n} v^{(k-n)}(0) = \binom{k}{n} n(n-1)(n-2) \cdots (n-(k-n)+1)(-b)^{k-n} \in \mathbb{Z}.$$

The result follows by (*). $\square$

Theorem. The number π is irrational.

Proof. Suppose $\pi = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ relatively prime. For fixed $n \geqslant 1$, define

$$F(x) = f(x) - f''(x) + f^{(4)}(x) - f^{(6)}(x) + \cdots + (-1)^n f^{(2n)}(x)$$

where $f(x) = \frac{1}{n!} x^n (a - bx)^n$ as above. Since

$$\frac{d}{dx}\big[F'(x)\sin x - F(x)\cos x\big] = [F''(x) + F(x)]\sin x = f(x)\sin x,$$

we have

$$\int_0^\pi f(x)\sin x\,dx = \Big[F'(x)\sin x - F(x)\cos x\Big]_0^\pi = F(0) - F(\pi) = F(0) - F\big(\tfrac{a}{b}\big) \in \mathbb{Z}.$$

On the interval $[0, \pi]$, the function $f(x) = \frac{1}{n!}\big(ax - bx^2\big)^n$ is maximized at the midpoint $\frac{a}{2b} = \frac{\pi}{2}$, so

$$0 < \int_0^\pi f(x)\sin x\,dx < \frac{\pi}{n!}\Big(\frac{\pi^2}{4}\Big)^n \to 0 \text{ as } n \to \infty.$$

For some $n \geqslant 1$, it follows that $0 < \int_0^\pi f(x)\sin x\,dx < 1$, contradicting the fact that the integral is an integer. $\square$

Theorem (Hermite, 1873). The number e is transcendental over $\mathbb{Q}$.

Proof. Suppose there exist $a_0, a_1, \ldots, a_m \in \mathbb{Z}$ such that

$$a_0 + a_1 e + a_2 e^2 + \cdots + a_m e^m = 0.$$

We may assume $a_0 a_m \neq 0$, and we seek a contradiction. Consider the polynomial

$$f(x) = \tfrac{1}{(p-1)!} x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p \in \mathbb{Q}[x]$$

of degree $mp + p - 1$ where p is a prime number larger than $\max\{m, |a_0|\}$ (but fixed for the moment). Note that for $0 < x < m$, we have

$$|f(x)| \leqslant \frac{m^{p-1}(m^p)^m}{(p-1)!} = \frac{m^{mp+p-1}}{(p-1)!}.$$

Following a trick due to Hurwitz, we define

$$F(x) = f(x) + f'(x) + f''(x) + \cdots + f^{(mp+p-1)}(x).$$

Since $f^{(mp+p)}(x) = 0$, we have

$$\frac{d}{dx}\left[e^{-x}F(x)\right] = [F'(x) - F(x)]e^{-x} = -e^{-x}f(x),$$

so

$$a_j e^j \int_0^j e^{-x} f(x)\,dx = -a_j e^j \left[e^{-x}F(x)\right]_0^j = a_j e^j F(0) - a_j F(j).$$

Summing over j gives

$$(\dagger) \qquad \sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)\,dx = -\sum_{j=0}^{m} a_j F(j) = -\sum_{j=0}^{m} \sum_{i=0}^{mp+p-1} a_j f^{(i)}(j).$$

Evidently, $f^{(i)}(j)$ is an integer divisible by p, unless $j = 0$ and $i = p - 1$:

- For $j \in \{1, 2, \ldots, m\}$, we factor $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{(p-1)!}(x-j)^p$ and $v(x) \in \mathbb{Z}[x]$. Since

  $$u^{(i)}(j) = \begin{cases} 0, & \text{for } i \neq p; \\ p, & \text{for } i = p, \end{cases}$$

  Leibniz' Formula gives $f^{(i)}(j) \in p\mathbb{Z}$ for all i in this case.

- For $j=0$, use the factorization $f(x)=u(x)v(x)$ where $u(x)=\frac{1}{(p-1)!}x^{p-1}$ and $v(x)\in\mathbb{Z}[x]$. In this case

$$u^{(i)}(0)=\begin{cases}0, & \text{for } i\neq p-1;\\ 1, & \text{for } i=p-1\end{cases}$$

and Leibniz' Formula gives

$$f^{(i)}(0)=\tbinom{i}{p-1}v^{(i-p+1)}(0)\in\mathbb{Z}.$$

Moreover, the binomial coefficient $\binom{i}{p-1}$ is divisible by p unless $i=p-1$, in which case we obtain

$$f^{(p-1)}(0)=v(0)=(-1)^p(-2)^p\cdots(-m)^p\ =\ \pm m!^p.$$

This integer is *not* divisible by p since we have chosen the prime $p>m$.

Now the right side of (†) is an integer congruent (mod p) to $-a_0f^{(p-1)}(0)=\mp a_0m!^p$, which is not divisible by p (by choice of the prime p). In particular,

$$(\ddagger)\qquad \sum_{j=0}^{m}a_je^j\int_0^j e^{-x}f(x)\,dx\ =\ N_p,\ \text{a nonzero integer.}$$

A contradiction follows by observing that the left side of (‡) tends to 0 for p sufficiently large:

$$\left|\int_0^j e^{-x}f(x)\,dx\right|\leqslant\int_0^\infty e^{-x}|f(x)|\,dx\ <\ \frac{m^{mp+p-1}}{(p-1)!}\ \to 0\ \text{ as } p\to\infty.\qquad\square$$

Before showing the transcendence of π, we recall the *elementary symmetric polynomials*

$$\begin{aligned}
s_0(x_1,x_2,\ldots,x_n)&=1;\\
s_1(x_1,x_2,\ldots,x_n)&=x_1+x_2+\cdots+x_n;\\
s_2(x_1,x_2,\ldots,x_n)&=\sum_{1\leqslant i<j\leqslant n}x_ix_j\ =\ x_1x_2+x_1x_3+\cdots+x_{n-1}x_n;\\
&\ \vdots\\
s_k(x_1,x_2,\ldots,x_n)&=\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n}x_{i_1}x_{i_2}\cdots x_{i_k};\\
&\ \vdots\\
s_n(x_1,x_2,\ldots,x_n)&=x_1x_2\cdots x_n.
\end{aligned}$$

Note that $s_k(x_1, x_2, \ldots, x_n)$ is a polynomial in $x_1, x_2, \ldots, x_n$ with $\binom{n}{k}$ terms, and that

$$(t + x_1)(t + x_2) \cdots (t + x_n) = \sum_{k=0}^{n} s_{n-k}(x_1, x_2, \ldots, x_k) t^k;$$

thus the coefficients in any monic polynomial are, up to $\pm$ signs, the elementary symmetric polynomials in the roots.

A polynomial $f(x_1, x_2, \ldots, x_n)$ is called *symmetric* if it is unchanged under arbitrary permutations of its n arguments; i.e. if

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)$$

for each $\sigma \in S_n$; here S_n is the group of all $n!$ permutations of $\{1, 2, 3, \ldots, n\}$ (i.e. bijections from the set $\{1, 2, \ldots, n\}$ to itself). Clearly each $s_k(x_1, x_2, \ldots, x_n)$ is symmetric in this sense, thereby justifying the name 'elementary symmetric polynomials'. More generally, every polynomial $P(s_1, s_2, \ldots, s_n)$ in the elementary symmetric polynomials $s_k = s_k(x_1, x_2, \ldots, x_n)$, with coefficients in $\mathbb{Q}$ (or in $\mathbb{Z}$) is symmetric in $x_1, \ldots, x_n$. We will require the converse of this statement: the *Fundamental Theorem of Invariant Theory*[2] (at least for the case of S_n permuting coordinates). This states that *every* symmetric polynomial $f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ (or in $\mathbb{Z}[x_1, x_2, \ldots, x_n]$) has the form

$$f(x_1, x_2, \ldots, x_n) = P(s_1, s_2, \ldots, s_n)$$

for some polynomial $P(t_1, \ldots, t_n) \in \mathbb{Q}[t_1, \ldots, t_n]$ (or in $\mathbb{Z}[t_1, \ldots, t_n]$, respectively). The proof is by straightforward induction on the degree, yet we omit it; and in lieu of a proof, we give a simple example for $n = 3$: The polynomial $x^3 + y^3 + z^3$ is symmetric in x, y, z, so it should be possible to write this as a polynomial in

$$s_1 = x+y+z, \quad s_2 = xy+xz+yz, \quad \text{and } s_3 = xyz$$

with integer coefficients. The desired expression is given by

$$s_1^3 - 3s_1s_2 + 3s_3 \;=\; (x+y+z)^3 - 3(x+y+z)(xy+xz+yz) + 3xyz \;=\; x^3+y^3+z^3.$$

[2] See e.g. P. Olver, *Classical Invariant Theory,* Cambridge Univ. Press, 1999, p.75.

Theorem (Lindemann, 1882). The number π is transcendental over $\mathbb{Q}$.

Proof. Suppose π is algebraic. Since $i = \sqrt{-1}$ is algebraic (of degree 2), it follows that πi is also algebraic; let $g_1(x) \in \mathbb{Q}[x]$ be its minimal polynomial, say of degree n. Over $\mathbb{C}$ we can factor

$$g_1(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$$

where $\alpha_1 = \pi i$. By the preceding remarks,

$$g_1(x) = x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)^{n-1} s_{n-1} x + (-1)^n s_n$$

where $s_1, s_2, \ldots, s_n$ are the elementary symmetric polynomials in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since $g_1(x) \in \mathbb{Q}[x]$, we have

$$s_1, s_2, \ldots, s_n \in \mathbb{Q}.$$

Denote $[n] := \{1, 2, \ldots, n\}$. For each subset $J = \{j_1, j_2, \ldots, j_m\} \subseteq [n]$ of size $|J| = m$, define

$$\alpha_J = \sum_{j \in J} \alpha_j = \alpha_{j_1} + \alpha_{j_2} + \ldots + \alpha_{j_m}.$$

For each $m \in [n]$, define

$$g_m(x) = \prod_{\substack{J \subseteq [n] \\ |J| = m}} (x - \alpha_J) \;=\; \prod_{1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant n} (x - \alpha_{j_1} - \alpha_{j_2} - \cdots - \alpha_{j_m}),$$

a polynomial of degree $\binom{n}{m}$. Note that for $m = 1$ we obtain the polynomial previously called $g_1(x)$, so our notation is consistent. Also the special case $m = n$ yields

$$g_n(x) \;=\; x - \alpha_1 - \alpha_2 - \cdots - \alpha_n \;=\; x - s_1.$$

Technically the case $m = 0$ should give $g_0(x) = 1$, but we really only need $g_1(x), \ldots, g_n(x)$.

If we now expand

$$g_m(x) \;=\; a_0 + a_1 x + a_2 x^2 + \cdots + a_{\binom{n}{m}} x^{\binom{n}{m}},$$

then each coefficient $a_\ell = a_{\ell,m}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is symmetric in $\alpha_1, \alpha_2, \ldots, \alpha_n$, since any permutation of the α_j's will also permute the $\binom{n}{m}$ subsets $J \subseteq [n]$ of size m, leaving the polynomial $g_m(x)$ unchanged. By the Fundamental Theorem of Invariant Theory, there exists a polynomial $P_{\ell,m}$ in n variables, with rational coefficients, such that

$$a_\ell = a_{\ell,m}(\alpha_1, \alpha_2, \ldots, \alpha_n) = P_{\ell,m}(s_1, s_2, \ldots, s_n).$$

However, $s_1, s_2, \ldots, s_n \in \mathbb{Q}$ as noted above; so $a_\ell \in \mathbb{Q}$ and we deduce that

$$g_m(x) \in \mathbb{Q}[x].$$

It may happen that $g_m(x)$ is divisible by x^ν; this will happen if there are ν subsets $J \subseteq [n]$ of size $|J| = m$ satisfying $\alpha_J \in \{0, \pm 2\pi i, \pm 4\pi i, \ldots\}$. We factor out these trivial factors, yielding

$$g_m(x) = x^\nu \tilde{g}_m(x) \quad \text{where } \tilde{g}_m(x) \in \mathbb{Q}[x],\ \tilde{g}_m(0) \neq 0.$$

Since $g_1(x)$ is irreducible, however, we have $\tilde{g}_1(x) = g_1(x)$. Next, define

$$g(x) = c\tilde{g}_1(x)\tilde{g}_2(x) \cdots \tilde{g}_n(x) \in \mathbb{Z}[x]$$

where c is the smallest positive integer for which this product has integer coefficients (i.e. c is the least common denominator of all coefficients in $\prod_m \tilde{g}_m(x) \in \mathbb{Q}[x]$). Because we have eliminated all factors x^ν, we have $g(0) \neq 0$. Note that

$$g(x) = c \prod_{\substack{J \subseteq [n] \\ \alpha_J \notin 2\pi i\mathbb{Z}}} (t - \alpha_J) \;=\; c(t - \beta_1)(t - \beta_2) \cdots (t - \beta_r)$$

where we have indexed the values of $\alpha_J \notin 2\pi i\mathbb{Z}$ as $\beta_1, \ldots, \beta_r$. Later we will also require the expansion

$$g(x) = cx^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0, \quad c_i \in \mathbb{Z},\ c_0 \neq 0.$$

By Euler's Formula, $e^{\pi i} + 1 = 0$ so

$$(1) \qquad (e^{\alpha_1} + 1)(e^{\alpha_2} + 1) \cdots (e^{\alpha_n} + 1) = 0.$$

Now expand (1) into 2^n terms by the distributive law. These terms are indexed by the 2^n subsets $J \subseteq [n]$, and a typical term has the form $\prod_{j \in J} e^{\alpha_j} = e^{\alpha_J}$. At least one such term (the constant term corresponding to $J = \varnothing$) is 1; let us say that there are exactly $k \geqslant 1$ terms equal to 1 in this sum (i.e. k subsets of $J \subseteq [n]$ for which $\alpha_j \in 2\pi i\mathbb{Z}$). The remaining terms $e^{\alpha_J} \neq 1$ are simply $e^{\beta_1}, e^{\beta_2}, \ldots, e^{\beta_r}$ with β_j as above; here $r = \deg g(x) = 2^n - k$. Now the expansion of (1) reads as

$$(2) \qquad e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_r} + k = 0,\ k \geqslant 1.$$

Define

$$f(x) = \frac{c^s x^{p-1} g(x)^p}{(p-1)!} \in \mathbb{Q}[x]$$

where $s = rp-1$ and p is a large prime number; and set

$$F(x) = f(x) + f'(x) + f''(x) + \cdots + f^{(s+p)}(x).$$

Since $\deg f(x) = rp + p - 1 = s + p$, we have $f^{(s+p+1)}(x) = 0$. Again using Hurwitz' trick,

$$\frac{d}{dx}\left[e^{-x}F(x)\right] = -e^{-x}f(x)$$

so

$$-\int_0^x e^{-y} f(y)\, dy \;=\; -F(0) + e^{-x}F(x).$$

Substituting $y = tx$ yields

$$-x\int_0^1 e^{(1-t)x} f(tx)\, dt \;=\; -e^x F(0) + F(x).$$

Evaluate at $x = \beta_1, \beta_2, \ldots, \beta_r$ and sum to get

$$\text{(3)} \qquad -\sum_{j=1}^{r} \beta_j \int_0^1 e^{(1-t)\beta_j} f(t\beta_j)\, dt \;=\; -(e^{\beta_1} + \cdots + e^{\beta_r})F(0) + \sum_{j=1}^{r} F(\beta_j)$$

$$= kF(0) + \sum_{j=1}^{r}\sum_{m=0}^{s+p} f^{(m)}(\beta_j).$$

Our strategy, as before, is to show that for any sufficiently large prime p, the right hand side of (3) is a nonzero integer; but the left side $\to 0$ as $p \to \infty$. To this end, we first claim that

$$\text{(4)} \qquad \sum_{j=1}^{r}\sum_{m=0}^{s+p} f^{(m)}(\beta_j) \text{ is an integer divisible by } p.$$

To see this, write $f(x) = pc^s h(x)$ where $h(x) = \frac{1}{p!}x^{p-1}g(x)^p$. If $m < p$, then the polynomial $h^{(m)}(x)$ is divisible by $g(x)^{p-m}$ and so $h^{(m)}(\beta_j) = 0$. Also since $h(x) = \frac{1}{p!}\sum_j a_j x^j$ where $a_j \in \mathbb{Z}$, we have $h^{(p)}(x) = \sum_j \binom{p+j}{j} a_{p+j} x^j \in \mathbb{Z}[x]$. Thus $h^{(m)}(x) \in \mathbb{Z}[x]$ for all $m \geqslant p$. Since $\sum_{j=1}^r h^{(m)}(\beta_j)$ is a symmetric polynomial of degree at most $rp-1 = s+p$ (assuming $m \geqslant p$) in $\beta_1, \ldots, \beta_r$ with integer coefficients, $\sum_{j=1}^r h^{(m)}(\beta_j)$ is a polynomial in $\frac{c_0}{c}, \frac{c_1}{c}, \ldots, \frac{c_{r-1}}{c}$ with integer coefficients. (The values $\frac{c_j}{c}$ are the coefficients in $\frac{1}{c}g(x) = \prod_j (x - \beta_j)$; hence the elementary symmetric polynomials in $\beta_1, \ldots, \beta_r$ take values $\pm\frac{c_0}{c}, \ldots, \pm\frac{c_{r-1}}{c}$. Here we have used the $\mathbb{Z}$-version of the Fundamental Theorem of Invariant Theory from p.6.) Since $\deg h^{(m)}(x) \leqslant s$ for $m \geqslant p$, the factor c^s clears all denominators to yield $\sum_{j=1}^r c^s h^{(m)}(\beta_j) \in \mathbb{Z}$. After multiplying by p and summing over m, we obtain (4).

Turning now to the constant $F(0)$ in (3), let us write $f(x) = u(x)v(x)$ where $u(x) = \frac{1}{(p-1)!}c^s x^{p-1}$ and $v(x) = g(x)^p$. Since

$$u^{(j)}(0) = \begin{cases} c^s, & \text{if } j = p-1; \\ 0, & \text{otherwise,} \end{cases}$$

by Leibniz' Formula we obtain

$$f^{(m)}(0) = \binom{m}{p-1} c^s v^{(j)}(0) = \binom{m}{p-1} \left[\frac{d^j}{dx^j} g(x)^p\right]_{x=0}.$$

This vanishes for $m \leqslant p-2$; and in general $v^{(j)}(0) \in \mathbb{Z}$ since $g(x) \in \mathbb{Z}[x]$. Also for $m \geqslant p$, the binomial coefficient $\binom{m}{p-1}$ is divisible by p, so

$$F(0) = \sum_{m=0}^{s+p} f^{(m)}(0) \;=\; c^s g(0)^p + pM_p \;=\; c^s c_0^p + pM_p$$

for some integer M_p.

Henceforth assume that the prime $p > \max\{k, c, c_0\}$. Since $kc^s c_0^p \neq 0$, it follows that the right side of (3) is an integer not divisible by p; in particular, the right side of (3) is nonzero.

In order to obtain a final contradiction, it remains only to show that the left side of (3) converges to 0 as the prime $p \to \infty$. We have

$$|f(t\beta_j)| \leqslant \frac{|c|^s |\beta_j|^{p-1} m_j^p}{(p-1)!}$$

where

$$m_j = \sup_{0 \leqslant t \leqslant 1} |g(t\beta_j)|.$$

Finally, if we let

$$B = \max_{1 \leqslant j \leqslant r} \left| \int_0^1 e^{(1-t)\beta_j}\, dt \right|,$$

then

$$\left| -\sum_{j=1}^{r} \beta_j \int_0^1 e^{(1-t)\beta_j} f(t\beta_j)\, dt \right| \leqslant \sum_{j=1}^{r} \frac{|\beta_j|^p |c|^s m_j^p B}{(p-1)!} \;=\; \frac{B}{|c|} \sum_{j=1}^{r} \frac{|\beta_j c^r m_j|^p}{(p-1)!} \;\to 0 \quad \text{as } p \to \infty,$$

the desired final contradiction. □