Changing the Coefficient Ring

1. Modules

Let R be a ring with identity. A *(left) R-module* is an additive abelian group M together with an operation of 'scalar multiplication' $R \times M \to M$ such that for all $r, s \in R$ and $x, y \in M$ we have

(i) $r(x + y) = rx + ry$;
(ii) $(r + s)x = rx + sx$;
(iii) $1x = x$;
(iv) $r(sx) = (rs)x$.

For example a module over a field $R = F$ is the same thing as a vector space over F. An additive abelian group is the same thing as a module over the ring $\mathbb{Z}$. For arbitrary R, the *free module of rank n over R* is a module isomorphic to $R^n = R \oplus R \oplus \cdots \oplus R$. This is a module over R with coordinatewise multiplication by R. However, if we consider R^n as consisting of all $n \times 1$ column vectors over R, then R^n it is also a module over the ring $R^{n\times n}$ of all $n \times n$ matrices over R.

We shall primarily be concerned with the case R is commutative. In this case there is no distinction between left and right R-modules. [In the case R is noncommutative, we must distinguish between left R-modules and right R-modules because of (iv); in a right R-module we have $(xr)s = x(rs)$. A right module over R is the same thing as a left module over the *opposite ring* R° which has the same elements and addition as R, but with multiplication defined by $r \circ s = sr$.]

A free module M over R is rather like a vector space in that it has a set of generators (which generate M by taking R-linear combinations), and a minimal generating set is rather like a basis: every minimal generating set has the same cardinality r which is the rank of M.

An arbitrary $\mathbb{Z}$-module (i.e. additive abelian group) G is a direct sum of cyclic groups, including possibly some infinite and some finite cyclic groups. Thus $G \cong \mathbb{Z}^r \oplus T(G)$ where $T(G)$ is the *torsion subgroup of* G, defined as the set of elements of G of finite order. Also r is called the *rank* of G. As a $\mathbb{Z}$-module, G is free (of rank r) iff $T(G) = 0$.

If M and N are R-modules then an *R-module homomorphism* from M to N is a map $f : M \to N$ satisfying $f(rx + r'x') = rf(x) + r'f(x')$ for all $r, r' \in R$ and $x, x' \in M$. The usual isomorphism theorems for groups and rings extend to modules; for example the First

Isomorphism Theorem gives $M/\ker f \cong f(M) \leq N$. The set $Hom_R(M,N)$ consisting of all R-module homomorphisms from M to N is itself an R-module. In particular we have the *dual R-module* $M^* = Hom(M,R)$. Also the collection of all endomorphisms of M, denoted $End(M) = End_R(M) = Hom_R(M,M)$, is an R-module; but more than this, since composition of elements of $End(M)$ is defined, this makes $End(M)$ a ring. It follows that $End(M)$ is in fact an R-algebra.

2. Tensor Products

Let R be a commutative ring with identity, and let M and N be R-modules. The *tensor product of M and N over R*, denoted $M \otimes_R N$ or simply $M \otimes N$, is defined as the quotient ring A/B where A is the free module generated by the symbols (x,y) where $x \in M$ and $y \in N$; and B is the submodule generated by the expressions

$$(rx + r'x', y) - r(x,y) - r'(x',y), \qquad (x, ry + r'y') - r(x,y) - r'(x,y')$$

where $r, r' \in R$; $x, x' \in M$; $y, y' \in N$. Note that A is a free module of rank $|M|\cdot|N|$, which is often infinite. The coset $(x,y) + B$ is denoted simply by $x \otimes y$. Informally, $M \otimes N$ is constructed by starting with R-linear combinations of the symbols $x \otimes y$ where $x \in M$ and $y \in N$, then imposing the bilinearity conditions

$$(rx + r'x') \otimes y = r(x \otimes y) + r'(x' \otimes y), \qquad x \otimes (ry + r'y') = r(x \otimes y) + r'(x \otimes y').$$

Note that the *only* identities that hold in $M \otimes N$, are those that are deducible from these bilinearity relations (just as the only relations that hold in a finitely presented group, are those that are deducible from the defining relations). In general the elements of $M \otimes N$ are not all of the form $x \otimes y$. The elements of this special form $x \otimes y$ are called *pure tensors*, and they generate $M \otimes N$ as an R-module.

2.1 Example Let $M = \mathbb{R}^3$, $N = \mathbb{C}$, $R = \mathbb{R}$. Consider the standard basis $\{e{=}(1,0,0), f{=}(0,1,0), g{=}(0,0,1)\}$ for $M = \mathbb{R}^3$, and the standard basis $\{1, i\}$ for $N = \mathbb{C}$ over $\mathbb{R}$. Then $M \otimes N$ is a 6-dimensional vector space over R with basis

$$\{e \otimes 1,\ e \otimes i,\ f \otimes 1,\ f \otimes i,\ g \otimes 1,\ g \otimes i\}.$$

A general element of $M \otimes N$ can be uniquely expressed in the form

$$\begin{aligned} &a_1(e \otimes 1) + a_2(e \otimes i) + a_3(f \otimes 1) + a_4(f \otimes i) + a_5(g \otimes 1) + a_6(g \otimes i) \\ &\qquad = (a_1e{+}a_3f{+}a_5g) \otimes 1 + (a_2e{+}a_4f{+}a_6g) \otimes i \\ &\qquad = e \otimes (a_1{+}a_2i) + f \otimes (a_3{+}a_4i) + g \otimes (a_5{+}a_6i) \end{aligned}$$

where $a_1, \ldots, a_6 \in \mathbb{R}$. The middle of these expressions shows how to uniquely decompose an arbitrary element of $\mathbb{R}^3 \otimes \mathbb{C}$ into its real part and its imaginary part, both of which are vectors in $\mathbb{R}^3$. The latter formulation shows that $\mathbb{R}^3 \otimes \mathbb{C}$ actually consists of all $\mathbb{C}$-linear combinations of the original basis vectors $e, f, g \in \mathbb{R}^3$, so that $\mathbb{R}^3 \otimes \mathbb{C}$ is the *complexification* of the real vector space $\mathbb{R}^3$.

The latter example admits several generalisations.

2.2 Example. Suppose M is an m-dimensional vector space over a field F, and let $E \supseteq F$ be an extension field of degree n (i.e. E has dimension n over F). Then $M \otimes_F E$ may be regarded not only as an mn-dimensional vector space over F, but also as an m-dimensional vector space over E. This is the most natural way to enlarge the field of coefficients of a vector space.

2.3 Example. Even more generally, if M and N are vector spaces of dimension m and n respectively, over a field F, then $M \otimes_F N$ is an mn-dimensional vector space over F. Indeed if $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_n\}$ are bases for M and N respectively, then $\{v_i \otimes w_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a basis for $M \otimes N$. This leads to our next example:

2.4 Example. Let M be the set of all $m \times 1$ column vectors over F, and let N be the set of all $1 \times n$ row vectors over F. Then $M \otimes_F N$ may be regarded as the set of all $m \times n$ matrices over F, in such a way that for all

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in M, \qquad y = [y_1,\ y_2,\ \ldots,\ y_n] \in N$$

we have

$$x \otimes y = xy = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix} \in M \otimes_F N.$$

Note that the matrix product xy is bilinear in both x and y as required. Also note that the pure tensors $x \otimes y$ are just the matrices of rank ≤ 1. If $r = \min\{m, n\}$ then every $m \times n$ matrix is expressible as a sum of r matrices of rank 1, so every vector in $M \otimes N$ is expressible as a sum of r pure tensors, and in general no fewer. The apparent lack of symmetry that asks us to write M as column vectors and N as row vectors, is explained in the next example.

2.5 Example. Let V and W be vector spaces of dimension m and n over a field F, respectively. Recall (Section 1) that $Hom(V, W) = Hom_F(V, W)$ denotes the set of all F-linear transformations $V \to W$; and $V^* = Hom_F(V, F)$ is the set of all F-linear transformations $V \to F$, i.e. linear functionals on V. After choosing bases for V and W, we may express vectors in V and W as row vectors of size $1 \times m$ and $1 \times n$ respectively; also elements of $Hom(V, W)$ are represented as $m \times n$ matrices over F. Linear functionals $f \in V^*$ are represented as $m \times 1$ matrices, i.e. column vectors over F. Indeed every linear functional $f \in V^*$ has the form

$$f(x) = [f_1,\ f_2,\ \ldots, f_m] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \sum_{1 \le i \le m} f_i x_i \in F$$

and so the matrix of $f : V \to F$ with respect to our basis for V, and the basis $\{1\}$ for F, is $[f_1,\ f_2,\ \ldots, f_m]$ where $f_i \in F$. Now it is evident that the isomorphism in Example 2.4 represents the identity $Hom_F(V, W) \cong V^* \otimes_F W$. A coordinate-free description of this isomorphism is given by mapping

$$V^* \times W \to Hom(V, W), \qquad (f, w) \mapsto f \otimes w$$

where $f \otimes w : V \to W$ is the linear transformation $v \mapsto f(v)w$.

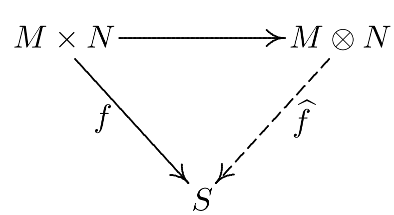
Why the need for the dual V^* in place of V in the latter example? One answer is that we can replace V^* by V if we are only interested in V as an abstract n-dimensional vector space over F, for then V^* is also n-dimensional over F and so $Hom(V, W) \cong V^* \otimes W \cong V \otimes W$. However, the isomorphism $V^* \cong V$ is not canonical (it requires first choosing bases for V and V^*, and there is no prefered choice of these bases). The comments in Example 2.5 show that the isomorphism $Hom(V, W) \cong V^* \otimes W$ is natural and canonical, while the isomorphism with $V \otimes W$ is not.

2.6 Example. Generalising the previous example, if R is any commutative ring with identity, and M and N are R-modules, then $Hom_R(M, N) \cong M^* \otimes N$.

A more useful way of understanding $M \otimes N$ than through the definition given above, is in terms of its universal property: Suppose S is an R-module and $f : M \times N \to S$ satisfies

$$f(rx + r'x', y) = rf(x, y) + r'f(x', y), \qquad f(x, ry + r'y') = rf(x, y) + r'f(x, y')$$

for all $r, r' \in R$; $x, x' \in M$; $y, y' \in N$. Then there exists a unique R-module homomorphism $\widehat{f} : M \otimes N \to S$ such that the following diagram commutes:



i.e. $f(x,y) = \widehat{f}(x \otimes y)$ for all $(x,y) \in M \times N$. This gives a way to interpret bilinear maps as linear maps: doing so however requires replacing the domain $M \times N$ by $M \otimes N$. Wherever tensor products are used, it is most often this universal property of tensor products that is important. In fact many books actually *define* tensor products in terms of this universal property, as follows. Let W be an R-module, and let $u : M \times N \to W$ be R-bilinear, i.e. $u(rx + r'x', y) = ru(x,y) + r'u(x', y)$ and similarly in the second argument. We call W (or more precisely the pair (W, u)) *a tensor product of* M *and* N if for every R-bilinear map $f : M \times N \to S$, there is a unique R-linear map $\widehat{f} : W \to S$ such that $f = \widehat{f} \circ u$. Now it is easy to see from this definition that if a tensor product of M and N exists then it is unique up to isomorphism. Then to prove existence one can use the quotient space construction given above.

2.7 Example. Let V and W be real vector spaces, and let $\widehat{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ and $\widehat{W} = W \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified vector spaces as in Examples 2.1, 2.2. (These are vector spaces over $\mathbb{C}$, such that $\dim_{\mathbb{C}}(\widehat{V}) = \dim_{\mathbb{R}}(V)$ and $\dim_{\mathbb{C}}(\widehat{V}) = \dim_{\mathbb{R}}(V)$). If $f : V \to W$ is a real linear transformation then there is a unique $\mathbb{C}$-linear map $\widehat{f} : \widehat{V} \to \widehat{W}$ extending f. This follows from the universal property as follows. First observe that $f : V \to W$ gives rise to a bilinear map

$$V \times \mathbb{C} \to \widehat{W}, \quad (v, \lambda) \mapsto f(v) \otimes \lambda.$$

By universality, there exists a unique $\mathbb{R}$-linear map $\widehat{f} : \widehat{V} \to \widehat{W}$ such that $\widehat{f}(v \otimes \lambda) = f(v) \otimes \lambda$ for all $\lambda \in \mathbb{C}$. But this property means that $\widehat{f}$ is $\mathbb{C}$-linear. And if $g : \widehat{V} \to \widehat{W}$ is any $\mathbb{C}$-linear map restricting to f, i.e. $g(v \otimes 1) = f(v) \otimes 1$, then $\mathbb{C}$-linearity implies that $g(v \otimes \lambda) = f(v) \otimes \lambda = \widehat{f}(v \otimes \lambda)$ for all $\lambda \in \mathbb{C}$. Since $\widehat{V}$ is the real span of the vectors $v \otimes \lambda$ for $v \in V$ and $\lambda \in \mathbb{C}$, this forces $g = \widehat{f}$.

Of course in the finite-dimensional case any matrix for f (with real entries) becomes a matrix for $\widehat{f}$ where we simply interpret the entries as complex numbers with zero imaginary part.

3. Computing Tor

Let M and N be R-modules, where R is a commutative ring with identity. The R-module $Tor(M, N)$ is defined on p.263 of the textbook using free resolutions. We will not explain this here. For now assume the formulas on p.265 for computing $Tor(M, N)$. To compute $Tor(M, N)$ whenever M and N are $\mathbb{Z}$-modules, it suffices to use formulas (1), (2) and (3), together with the identity

$$Tor(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z} \qquad \text{where } g = gcd(m, n)$$

which follows from (5). For example

$$\begin{aligned}
&Tor\big(\mathbb{Z}\oplus(\mathbb{Z}/4\mathbb{Z}), (\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z})\big)\\
&\qquad\cong Tor\big(\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z})\big) \oplus Tor\big(\mathbb{Z}/4\mathbb{Z},\ (\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z})\big)\\
&\qquad\cong 0 \oplus Tor(\mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z}) \oplus Tor(\mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z})\\
&\qquad\cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}).
\end{aligned}$$

4. Exact Sequences

A sequence

$$\cdots \xrightarrow{f_{n+2}} C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots$$

(where the C_i's are R-modules and each f_i is an R-module homomorphism) is *exact* if, for every n, the image $f_{n+1}(C_{n+1})$ coincides with the kernel of $f_n : C_n \to C_{n-1}$. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

This implies that f is injective, and so we may identify A with the submodule $f(A) \subseteq B$; also g is surjective, so the First Isomorphism Theorem gives

$$C \cong B/A \cong B/f(A).$$

The latter isomorphism is *equivalent to* the existence of the short exact sequence given above. Given such an isomorphism, there may or may not exist a submodule $U \subseteq B$ complementary to A, i.e. satisfying $B = U \oplus A$; but if such a submodule exists then we must have $U \cong B/A \cong C$. In this case we say that the exact sequence *splits.* Note that even if such a complementary submodule $U \subseteq B$ exists, it need not be unique, nor is the choice of such a complementary submodule U natural or canonical. For example the short exact sequence of $\mathbb{Z}$-modules (i.e. additive abelian groups) given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

in which the arrow labeled '2' is the map $x \mapsto 2x$, is not split. Every exact sequence of vector spaces is split, although the choice of complementary subspace is not unique in general.

5. Homology with Coefficients

Let R be a commutative ring with identity, and let X be a topological space with a simplicial subdivision. We denote by $C_n(X; R)$ the set of n-chains of a topological space

X with coefficients in R, i.e. the free R-module generated by the n-simplices. Then $C_n(X;R) = C_n(X) \otimes_{\mathbb{Z}} R$ where $C_n(X) = C_n(X;\mathbb{Z})$. The boundary maps ∂ are more than homomorphisms of additive abeliean groups; they are R-module homomorphisms

$$\cdots \xrightarrow{\partial} C_3(X;R) \xrightarrow{\partial} C_2(X;R) \xrightarrow{\partial} C_1(X;R) \xrightarrow{\partial} C_0(X;R) \longrightarrow 0.$$

Denote by $Z_n(X;R)$ the set of n-cycles, i.e. the kernel of $\partial : C_n(X;R) \to C_{n-1}(X;R)$. Also denote by $B_n(X;R)$ the set of n-boundaries, i.e. the image $\partial(C_{n+1}(X;R)) \le C_n(X;R)$. The n-th homology group of X with coefficients in R is the quotient group $H_n(X;R) = Z_n(X;R)/B_n(X;R)$. The homology groups with coefficients in $\mathbb{Z}$ (the default) determine the homology groups with coefficients in the arbitrary ring R; however the first guess that $H_n(X;R) = H_n(X) \otimes R$ (where $H_n(X) = H_n(X;\mathbb{Z})$) is not quite correct. The correct answer is given by

Universal Coefficient Theorem for Homology. *We have a split exact sequence*

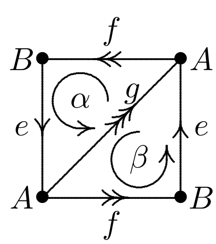
$$0 \longrightarrow H_n(X) \otimes R \longrightarrow H_n(X;R) \longrightarrow Tor(H_{n-1}(X), R) \longrightarrow 0.$$

Consequently

$$H_n(X;R) \cong \big(H_n(X) \otimes R\big) \oplus Tor(H_{n-1}(X), R)$$

(although there is no natural or canonical choice of complementary submodule isomorphic to $Tor(H_{n-1}(X), R)$).

The statement of this result in the textbook (p.264) is somewhat more general in that the coefficient ring R is replaced by an arbitrary additive abelian group G. My choice to state these results using a ring R is not intended to limit you, but rather to encourage you to think more concretely of rings such as $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ as typically arise in practice. Note that if $R = F$ is a field then $H_n(X;F) \cong F^r$ where $r = \dim_F(Z_n(X;F)) - \dim_F(B_n(X;F))$ and these vector space dimensions are easily computed by linear algebra.



5.1 Example. Let $X = P^2\mathbb{R}$, the real projective plane. We use the triangulation of X given in the previous handout; see the figure above and recall that

$$H_2(X) = H_2(X;\mathbb{Z}) = 0;$$
$$H_1(X) = H_1(X;\mathbb{Z}) \cong \mathbb{F}_2 \cong \mathbb{Z}/2\mathbb{Z};$$
$$H_0(X) = H_0(X;\mathbb{Z}) \cong \mathbb{Z}.$$

Here $\mathbb{F}_2 = \{0, 1\}$ is the field of order 2. The Universal Coefficient Theorem gives

$$\begin{array}{lcl} H_2(X;\mathbb{R}) = 0; & & H_2(X;\mathbb{F}_2) \cong \mathbb{F}_2; \\ H_1(X;\mathbb{R}) = 0; & \text{and} & H_1(X;\mathbb{F}_2) \cong \mathbb{F}_2; \\ H_0(X;\mathbb{R}) \cong \mathbb{R} & & H_0(X;\mathbb{F}_2) \cong \mathbb{F}_2\,. \end{array}$$

These groups may be computed directly as follows: We have $H_2(X;R) = Z_2(X;R)/\langle 0\rangle \cong Z_2(X;R)$ which is the set of all $k\alpha + \ell\beta$ where $k, \ell \in R$ such that

$$0 = \partial(k\alpha + \ell\beta) = k(e+f+g) + \ell(e+f-g) = (k+\ell)(e+f) + (k-\ell)g\,.$$

If $R = \mathbb{Z}$ or $\mathbb{R}$ then $k+\ell = k-\ell = 0$ implies $k = \ell = 0$ and so $H_2(X;R) = 0$. However, if $R = \mathbb{F}_2$ then the solutions are given by $k = \ell \in \mathbb{F}_2$ so $H_2(X;\mathbb{F}_2) \cong \langle \alpha+\beta\rangle_{\mathbb{F}_2} \cong \mathbb{F}_2$.

Writing $u = e + f \in C_1(X;R)$ we have $Z_1(X;R) = \langle u, g\rangle_R$ and $B_1(X;R) = \langle u+g, u-g\rangle_R = \langle u+g, 2g\rangle_R \le Z_1(X;R)$. For $R = \mathbb{Z}$ this gives $H_1(X) = \{B_1,\, g+B_1\} \cong \mathbb{F}_2$. For $R = \mathbb{R}$ we have $B_1(X;\mathbb{R}) = Z_1(X;\mathbb{R}) = \langle u, g\rangle_{\mathbb{R}} \cong \mathbb{R}^2$ and $H_1(X;\mathbb{R}) = 0$. For $R = \mathbb{F}_2$ we have $Z_1(X;\mathbb{F}_2) = \langle u, g\rangle_{\mathbb{F}_2} \cong \mathbb{F}^2$ and $B_1(X;\mathbb{F}_2) = \langle u+g\rangle_{\mathbb{F}_2} \cong \mathbb{F}_2$ so subtracting dimensions gives $H_1(X;\mathbb{F}_2) \cong \mathbb{F}_2$.

Also $H_0(X;R) = Z_0(X;R)/B_0(X;R) = \langle A, B\rangle_R/\langle A-B\rangle_R$. If $R = \mathbb{Z}$ then we have $H_0(X;\mathbb{Z}) = \{kA+B_0 : k \in \mathbb{Z}\} \cong \mathbb{Z}$. If $R = F$ is any field (such as $\mathbb{R}$ or $\mathbb{F}_2$) then $Z_0(X;F) = \langle A, B\rangle_F \cong F^2$ and $B_0(X;F) = \langle A-B\rangle_F \cong F$ so subtracting dimensions gives $H_0(X;F) \cong F$.