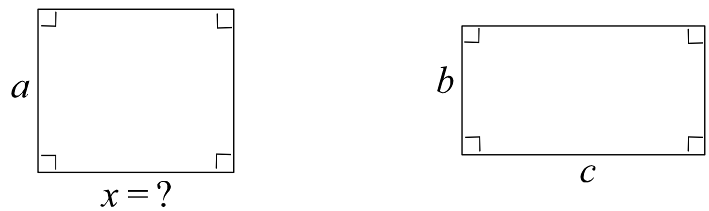
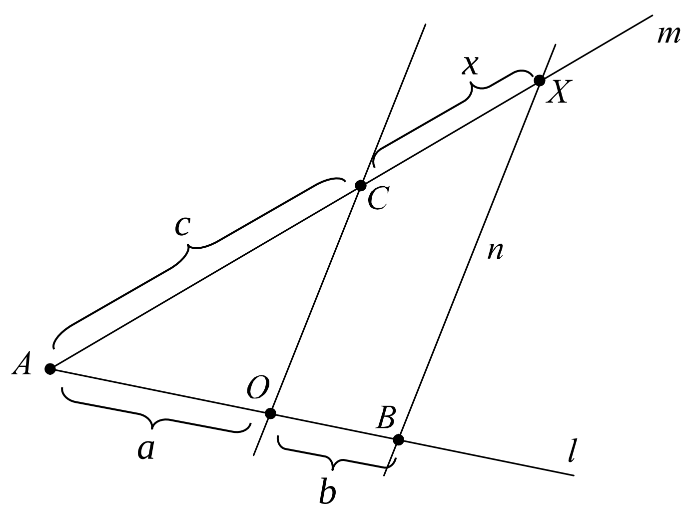
More Straightedge and Compass Constructions

Consider the following construction problem: You are given a rectangle with sides b and c, and you are asked how long to make a second rectangle with width a, so that its area equals that of the first rectangle. Of course we are being asked to solve the equation $ax = bc$ for the unknown distance x; but rather than solve this equation algebraically (which is trivial) we might ask rather to construct a line segment of the required length x, using straightedge and compass, given line segments of lengths a, b and c.



A solution of this problem is given as follows.

Lemma 1. Given distances a, b, c, one may construct (using straightedge and compass) a distance x such that $ax = bc$.



Proof. Use the compass to mark points O, A and B on a line l such that $OA = a$ and $OB = b$, as shown. Draw a second line m through A. (The choice of angle between l and m is arbitrary.) Locate a point C on m such that $AC = c$. Construct the unique line n through B parallel to OC. Let X be the point of intersection of m and n. The triangles AOC and ABX are similar (since corresponding angles are equal) so corresponding sides are in the same proportion; in particular

$$\frac{a}{c} = \frac{AO}{AC} = \frac{AB}{AX} = \frac{OB}{CX} = \frac{b}{x}$$

where $x = CX$. So this gives a construction of the required distance. □

Lemma 2. Given two disjoint circles C_1 and C_2 which are not concentric, one may construct a circle C that inverts the first two circles to two concentric circles C_1' and C_2'.

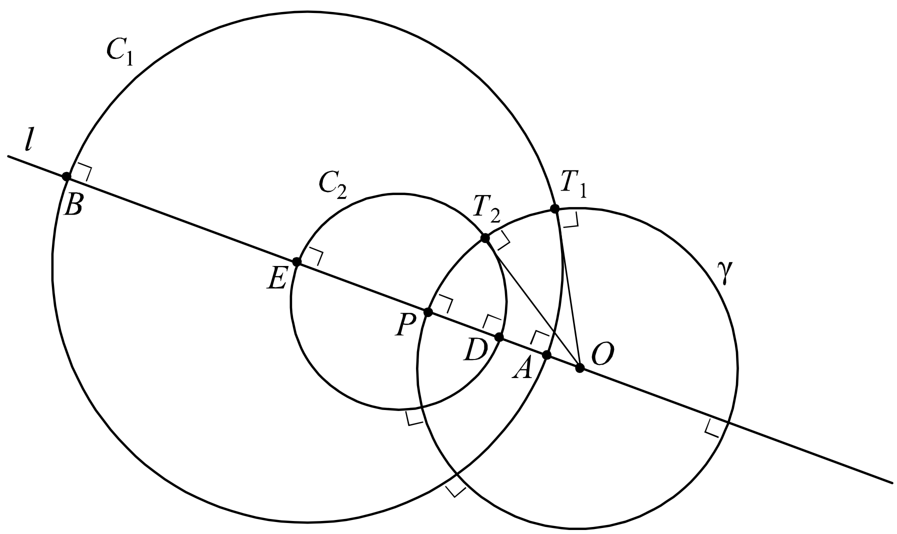
Proof. Let l be the line joining the centers of C_1 and C_2. (To construct l we must first find the centers of C_1 and C_2, assuming these are not already known; this is done as in HW3. By assumption these two centers are distinct, and so l is the line joining them.) By symmetry, the circle C should have center lying on the line l; but where? Let A and B be the points of intersection of l with C_1; and D and E are the points of intersection of l with C_2, labeled in the order shown. (We have pictured only the case that C_2 lies inside C_1; but the cases where C_1 lies inside C_2, or neither circle lies inside the other, may be solved in a similar fashion; we leave this as an exercise.) We first construct a circle γ orthogonal to l, C_1 and C_2. Such a circle has center O on l, and radius $r = OT_1 = OT_2$ where T_1 and T_2 are points of contact of tangent lines to C_1 and C_2 passing through O.

The position of the point O on the line l is determined by the length $x = OA$. Note that the points A and B are inverse with respect to γ; so also the points D and E. Therefore

$$OA \cdot OB = r^2 = OD \cdot OE ,$$

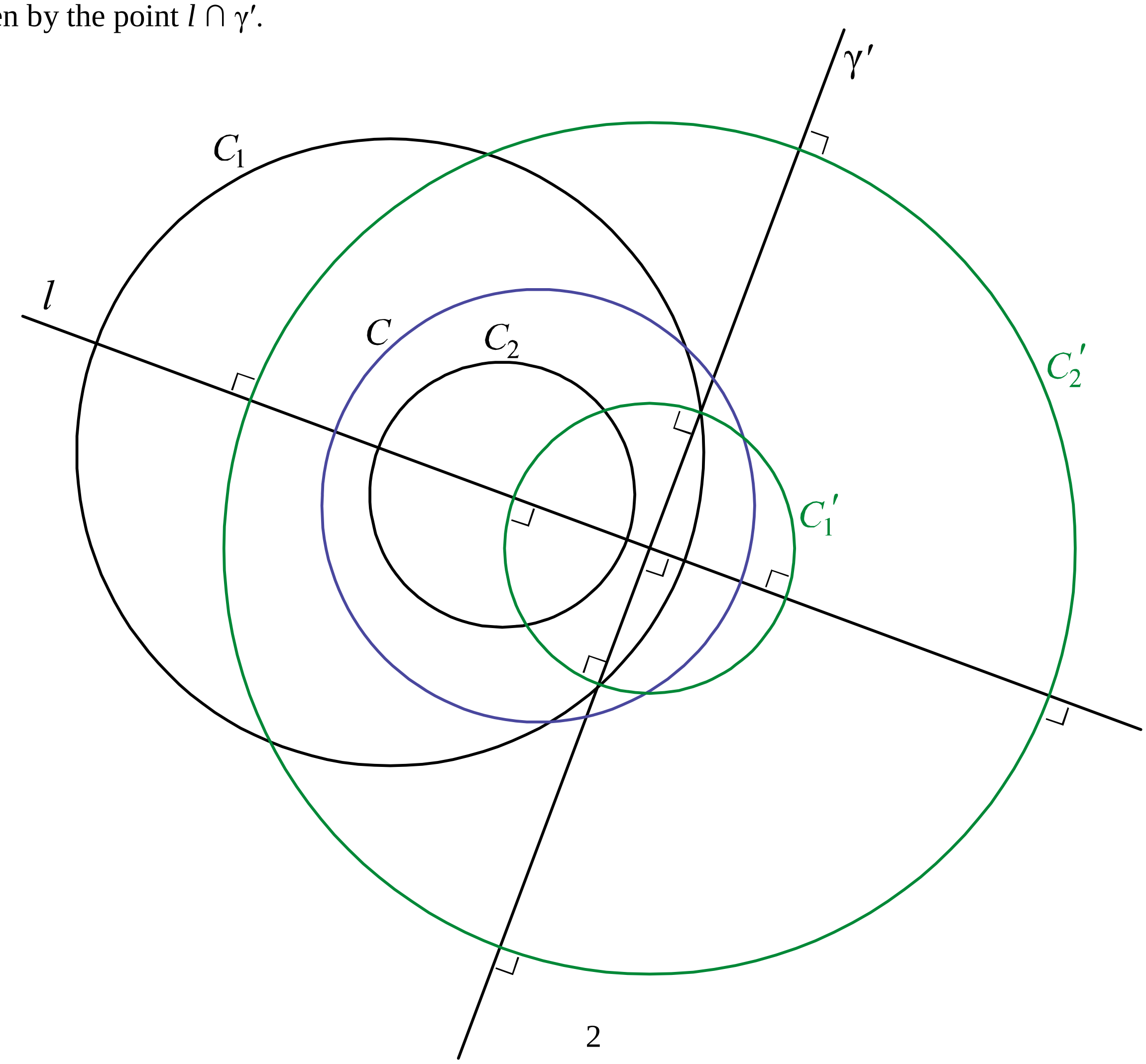
i.e.

$$x(AB + x) = (AD + x)(AE + x).$$



Fortunately the x^2 terms cancel, leaving us with the relation $ax = AD \cdot AE$ to solve for x, where a = AB – AD – AE = BE – AD. By Lemma 1 we may construct a line segment of the required length x and therefore locate the center O of the circle γ. To find the radius of γ we require only the radius $r = OT_1 = OT_2$; the construction of the length of a tangent from a given point O to a given circle (C_1 or C_2) has been described previously.

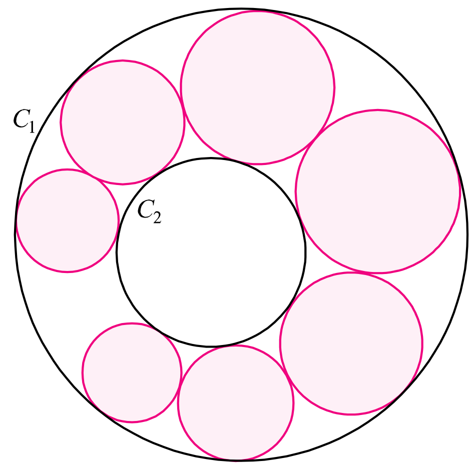
Now let P be one of the points of intersection of l with γ, and let C be any circle centered at P. Then inversion in C must take l to l, and γ to a line γ'; and it must take C_1 and C_2 to two circles C_1' and C_2', both of which are orthogonal to l and to γ'. Therefore C_1' and C_2' both have center given by the point $l \cap \gamma'$. □

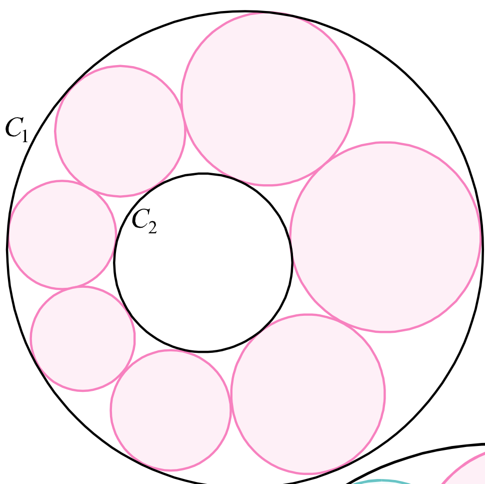


Similarly one may ask: if the circles C_1 and C_2 meet, to what extent can the picture be simplified using an inversion? If C_1 and C_2 are tangent at a point P, then by inverting with respect to a circle centered at P, the circles C_1 and C_2 are sent to a pair of parallel lines. And if C_1 and C_2 meet at two distinct points P and Q, then by inverting with respect to a circle centered at P, the circles C_1 and C_2 are sent to a pair of intersecting lines.

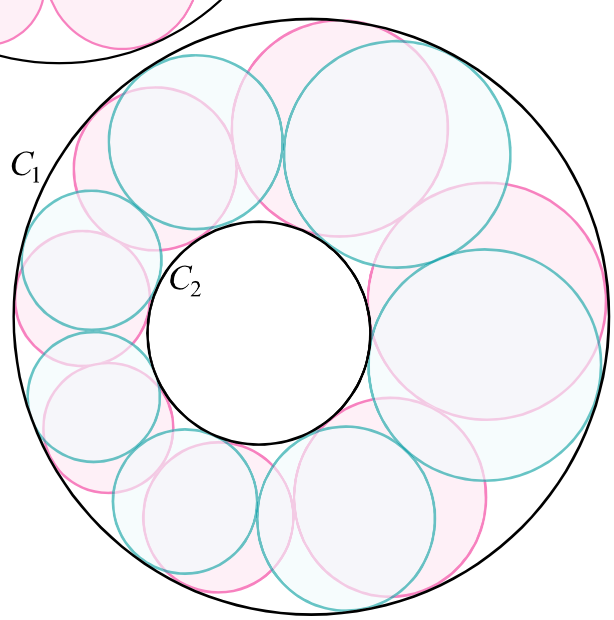
Steiner's Porism

An example of how inversion can be used to simplify a geometric problem, is the following. Consider non-intersecting circles C_1 and C_2, and a chain of circles 'between' C_1 and C_2 as shown: each circle in the ring is tangent to both C_1 and C_2; moreover adjacent circles in the chain are also tangent to each other. In the figure shown at the right, it is not possible to complete the chain of seven circles to a complete ring (the first and last circles in the chain do not tough, and there is not enough space in between to fit an eighth circle).

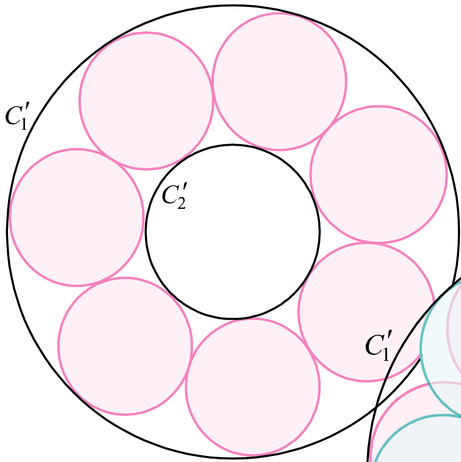




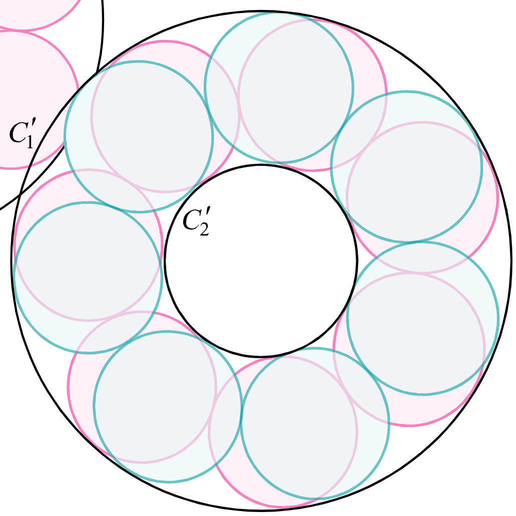
In the next figure at the left I have reduced the radius of C_2 just enough so that the seven intermediate circles in the chain become slightly larger and the first and last circles in the chain just manage to touch, forming a closed ring of seven circles. Of course by changing the size or position of the circles C_1 and C_2, we may obtain similar pictures with rings of any number of circles exceeding two.



What is not clear is: does the position of the first circle in the ring matter? i.e. if the first circle in the ring were placed in a different position (but still tangent to both C_1 and C_2 as before), and constructing the ring of circles between C_1 and C_2 as before, will we still obtain a ring with the same number of circles? The figure at the right suggests that the answer is yes; but why?



One case in which the answer to this question is obviously ‘yes’ is the special case of two circles C_1' and C_2' which are concentric (i.e. they have the same center) for then if the radii of C_1' and C_2' are such that there exists a ring of circles between C_1' and C_2' (as shown in the figure at the left) then clearly the position of the first circle in the ring is not critical; any other position is obtained from the first by a rotation about the common center of C_1' and C_2', and this rotation will transform one ring of intermediate circles to another (as shown on the right) with the same number of circles in the ring. (Think of the ball bearings in the hub of a bicycle wheel, if you are familiar with their design.) What inversion shows us, is that if this property holds for concentric circles, then it holds also in the general case considered above (arbitrary non-intersecting circles C_1 and C_2). This is because by Lemma 2, we can perform an inversion that sends the circles C_1 and C_2 to concentric circles C_1' and C_2'; and since inversion takes circles to circles, it will take any chain of intermediate circles between C_1 and C_2, to a similar chain of intermediate circles between C_1' and C_2', and conversely.



Thus the answer to the question on the previous page is ‘yes’: the position of the first circle in the chain of intermediate circles does not affect whether the chain completes to a ring, or how many intermediate circles constitute this ring.

A live demonstration of this phenomenon is available using the applet found at
http://members.ozemail.com.au/~llan/steiner.html
where it is possible to choose the number of circles in the ring.