Separation Properties

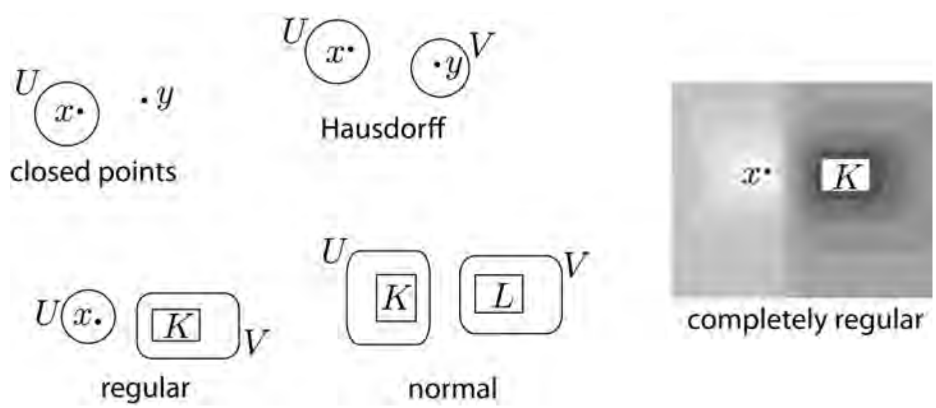
(November, 2015 version)

G. Eric Moorhouse, University of Wyoming

1. Separation Axioms

Let X be a topological space. We say that

- Points in X are closed if for every pair of points $x \neq y$, x has an open neighbourhood not containing y.
- X is **Hausdorff** if every pair of points $x \neq y$ has a pair of disjoint open neighbourhoods ($x \in U$, $y \in V$, $U \cap V = \varnothing$).
- X is **regular** if for every closed set $K \subset X$ and every point $x \notin K$, there are disjoint open sets $U, V \subset X$ with $x \in U$ and $K \subseteq V$.
- X is **normal** if for every pair of disjoint closed sets $K, L \subset X$, there are disjoint open sets $U, V \subset X$ with $K \subseteq U$ and $L \subseteq V$.
- X is **completely regular** if for every closed set $K \subset X$ and every point $x \notin K$, there exist a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \in K$.



Many properties similar to these are considered in the literature—we aim not for completeness, but just to list some of the most common and most important of these properties. Care is needed in reading the mathematical literature, as some authors preface their work saying that they only consider those topological spaces satisfying certain of these properties, effectively adding these properties as additional axioms to those we have used to define a topology. Other designations such as T_2, T_3, etc. are often used in place of the words we have used. A reasonable overview of these separation axioms, and an explanation of which combinations of which properties imply which others, can be found in [SS]… *but* *be warned:* the terminology, and the designations T_i, are used inconsistently throughout much of the literature, including [SS].

Here are a few facts about these separation properties that we will use.

1.1 Proposition. Every compact Hausdorff space has closed points, and is both regular and normal.

Proof. Let X be a compact Hausdorff space. It is clear that points in X are closed. Let $K \subset X$ be closed and suppose $x \notin K$. For every point $y \in K$, choose disjoint open sets U_y and V_y with $x \in U_y$ and $y \in V_y$. Note that $K \subseteq \bigcup_{y \in K} V_y$. Since K is a closed subset of a compact space, K is compact; so

$$K \subseteq V := V_{y_1} \cup V_{y_2} \cup \cdots \cup V_{y_n}$$

for some $y_1, y_2, \ldots, y_n$; and

$$x \in U := U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n} .$$

Moreover U and V are disjoint open sets, so X is normal.

Now let $K, L \subset X$ be disjoint open sets. Since X is regular, for each $x \in K$ there exist disjoint open sets $U_x, V_x \subset X$ such that $x \in U_x$ and $L \subseteq V_x$. Since the sets V_x form an open cover of the compact set L, there exist $x_1, x_2, \ldots, x_m \in K$ such that

$$L \subseteq V := V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_m} .$$

Also

$$K \subseteq U := U_{x_1} \cap U_{x_2} \cap \cdots \cap U_{x_m} .$$

Since U and V are disjoint open sets, X is normal. □

Not every subspace of a normal space is normal; but we will soon see (using Urysohn's Lemma) that every subspace of a normal space is completely regular. Here are some easier results of this nature:

1.2 Proposition.
(a) Every closed subspace of a normal space is normal.
(b) Every completely regular space is regular.
(c) Every subspace of a completely regular space is completely regular.

Proof. (a) Consider a closed subset $X \subseteq Y$ where Y is normal. Note that a subset $K \subseteq X$ is closed in X, iff it is closed in Y. (To see this, observe that a closed set in X has the form $K = K' \cap X$ where K' is closed in Y; but then K is also closed in Y.) So given two disjoint closed subsets $K, L \subset X$, there exist disjoint open sets $U, V \subset Y$ such that $K \subseteq U$ and $L \subseteq V$. Now $U \cap X$ and $V \cap X$ are disjoint sets, open in X, which separate K and L.

(b) Suppose X is completely regular. Consider a closed set $K \subset X$ and a point $x \notin K$. There exists a continuous map $f : X \to [0,1]$ such that f takes values 0 and 1 on x and K respectively. Then $U = f^{-1}([0, \frac{1}{3}))$ and $V = f^{-1}((\frac{2}{3}, 1])$ are disjoint open sets separating x from K.

(c) Let $X \subseteq Y$ where Y is completely regular. Let $K \subset X$ be closed in X, so that $K = K' \cap X$ for some closed set $K' \subset Y$. If $x \in X \smallsetminus K$ then $x \notin K'$, so there exists a continuous function $f : Y \to [0,1]$ separating x from K'. The restriction $f'|_X$ is a continuous function $X \to [0,1]$ separating x from K, so X is completely regular. □

It is also easy to prove that every metric space is Hausdorff, regular and normal. In fact if X is a metric space with points $x \neq y$, let $\delta = d(x,y) > 0$; then $B_{\delta/2}(x)$ and $B_{\delta/2}(y)$ are disjoint open neighbourhoods separating x and y, so X is Hausdorff. It is easy to see that the corresponding distance function $d : X \times X \to [0, \infty)$ is continuous. We may define the distance from a point $x \in X$ to an arbitrary subset $S \subseteq X$ by

$$d(x, S) = \inf_{y \in S} d(x, y);$$

and then it is easy to show that the map $X \to [0, \infty)$ defined by $x \mapsto d(x, S)$ is continuous; and $d(x, S) = 0$ iff $x \in \overline{S}$. Now if K and L are disjoint closed subsets of X, then the function $f : X \to [0,1]$

$$f(x) = \frac{d(x,K)}{d(x,K) + d(x,L)}$$

is a continuous function separating K from L; it takes values 0 and 1 on K and on L respectively.

2. Urysohn's Lemma

Pavel Urysohn
(1898–1924)

Informally, a topological space is normal iff any two disjoint closed sets can be separated by disjoint open sets. We may ask for an apparently stronger condition, namely that any two disjoint sets K, L are *separated by a continuous function* $f : X \to [0,1]$ if $f(x) = 0$ for all $x \in K$, and and $f(y) = 0$ for all $y \in L$. However, this turns out to be equivalent*:

* This is in contrast to the fact that not every regular space is completely regular. While there *is* a notion of completely normal spaces, it evidently does *not* mean that disjoint closed sets can be separated by a function. (A space is **completely normal** if every subspace is normal.)

2.1 Theorem (Urysohn's Lemma). Let X be a topological space. Then X is normal iff any two disjoint closed sets can be separated by a continuous function $X \to [0,1]$.

The proof we give is a hybrid of Munkres' proof with the usual proof found in other sources. In preparation, we observe the following equivalent formulation of normality:

2.2 Lemma. Let K be a closed subset of a normal space X, and suppose $K \subseteq V$ for some open set $V \subseteq X$. Then there exists an open set U such that

$$K \subseteq U \subseteq \overline{U} \subseteq V.$$

Proof. Since K and $X \smallsetminus V$ are disjoint open sets, using normality we get disjoint open sets U, V_1 such that $K \subseteq U$ and $X \smallsetminus V \subseteq V_1$; then $V \supseteq X \smallsetminus V_1 \supseteq \overline{U}$. $\square$

Proof of Theorem 2.1. Let K, L be disjoint closed sets in X. Suppose there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for all $x \in K$, and $f(y) = 1$ for all $y \in L$. Then $U = f^{-1}\big([0, \frac{1}{3})\big)$ and $V = f^{-1}\big((\frac{2}{3}, 1]\big)$ are disjoint open sets separating K from L.

Conversely, suppose X is normal, and let $K, L \subseteq X$ be disjoint closed sets. Let $U_1 = X \smallsetminus L$. By Lemma 2.2 we choose an open set U_0 with

$$K \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1 = X \smallsetminus L.$$

Similarly, we find an open set $U_{\frac{1}{2}}$ with

$$K \subseteq U_0 \subseteq \overline{U_0} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_1 = X \smallsetminus L.$$

Next, we find open sets $U_{\frac{1}{4}}, U_{\frac{3}{4}}$ such that

$$K \subseteq U_0 \subseteq \overline{U_0} \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq U_1 = X \smallsetminus L.$$

Continuing in this way, we find a family of open sets U_r indexed by the dyadic rationals $r \in [0,1]$ (i.e. all rationals in $[0,1]$ whose denominator is a power of 2) such that

$$K \subseteq U_0 \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq U_1 = X \smallsetminus L$$

whenever $r < s$ where $r, s \in R := \{\text{dyadic rationals in } [0,1]\}$. Now define $f : X \to [0,1]$ by

$$f(x) = \inf\{r \in R : x \in U_r\}.$$

Clearly $f(x) = 0$ for all $x \in K$, and $f(x) = 1$ for all $x \in L$. To see that f is continuous, let $\mathbb{I}$ denote the set of all *irrationals* in $(0, 1)$, and note that the sets of the form

$$[0, a),\ (a, 1] \text{ where } a \in \mathbb{I}$$

form a subbasis for the open sets of $[0, 1]$. Now for $x \in X$ and $a \in \mathbb{I}$ we have

$$f(x) < a \text{ iff } x \in U_r \text{ for some } r \in R \cap [0, a)$$

so $f^{-1}([0, a)) = \bigcup\{U_r : r \in R \cap [0, a)\}$ which is open in X; also

$$\begin{aligned} f(x) > a &\text{ iff } x \notin U_r \text{ for some } r \in R \cap (a, 1] \\ &\text{ iff } x \notin \overline{U_s} \text{ for some } s \in R \cap (a, 1] \end{aligned}$$

(take $r \in R \cap (f(x), a)$ and $s \in R \cap (a, r)$) so that $f^{-1}((a, 1]) = \bigcup\{X \setminus \overline{U_s} : s \in R \cap (a, 1]\}$ which is open in X. Thus f is continuous. □

3.2 Corollary. Every normal Hausdorff space is completely regular.

Urysohn's Lemma has the following generalization:

3.3 Theorem (Tietze-Urysohn Extension Theorem). Suppose X is a normal space, and let $K \subseteq X$ be a closed subset. Then every continuous function $f : K \to \mathbb{R}$ extends to a continuous function $X \to \mathbb{R}$. Moreover if f is bounded, then it has a bounded continuous extension to X.

Note that in the hypotheses of Urysohn's Lemma, if we define $f : K \sqcup L \to [0, 1]$ having values 0 and 1 on K and on L respectively, then f is continuous; so the extension provided by Theorem 3.3 gives another proof of Urysohn's Lemma. However, we omit the proof of Theorem 3.3.

4. Subspaces of Cubes

Tychonoff characterized the subspaces of cubes $[0, 1]^A$ (arbitrary products of the unit interval) as precisely the completely regular Hausdorff spaces:

4.1 Theorem (Tychonoff). Let X be a topological space. Then X is embeddable in $[0, 1]^A$ for some set A, iff X is completely regular and Hausdorff.

Proof. Since $[0, 1]^A$ is compact Hausdorff, it is normal and hence completely regular; thus every subspace of $[0, 1]^A$ is completely regular and Hausdorff.

Conversely, suppose X is completely regular. Denote by F the set of all continuous functions $X \to [0,1]$. We will embed X in $[0,1]^F = \prod_{f \in F} [0,1]_f$ where $[0,1]_f \simeq [0,1]$ for all f. For this we define the map

$$\iota : X \to [0,1]^F, \quad x \mapsto \big(f(x)\big)_{f \in F} \in [0,1]^F;$$

thus the f-coordinate of $\iota(x)$ is $f(x)$. To see that ι is injective, note that for distinct points $x, y \in F$, by complete regularity there exists $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$; and $\iota(x) \neq \iota(y)$ since these two points have f-coordinate 0 and 1 respectively.

Consider a subbasic open set in $[0,1]^F$ of the form

$$\pi_g^{-1}(V) = \big\{ v {=} (v_f)_f \in [0,1]^F \;:\; v_g \in V \big\}$$

where $g \in F$ and $V \subseteq [0,1]$ is open; then

$$\iota^{-1}(\pi_g^{-1}(V)) = \big\{ x \in X \;:\; g(x) \in V \big\} = g^{-1}(V),$$

which is open in X. This shows that $\iota : X \to [0,1]^F$ is continuous.

Let $U \subseteq X$ be an open neighbourhood of x, so that $\iota(x) \in \iota(U)$. Since X is completely regular, there exists a continuous function $g : X \to [0,1]$ such that $g(x) = 0$ and $g(y) = 1$ for all $y \notin U$. Now

$$V = \pi_g^{-1}\big(\big[0, \tfrac{1}{2}\big)\big) = \big\{ (v_f)_f \in [0,1]^F \;:\; v_g < \tfrac{1}{2} \big\}$$

is a (subbasic) open neighbourhood of $\iota(x)$ in $[0,1]^F$, and so $V \cap \iota(X) \subseteq \iota(U)$ is an open neighbourhood of $\iota(x)$ in the subspace topology for $\iota(X)$. This completes our proof that the restriction $\iota : X \to \iota(X)$ is a homeomorphism. □

We remark that the closed subspace $\overline{\iota(X)} \subseteq [0,1]^F$ in Theorem 4.1 is a compact Hausdorff space in which X is embedded. This is in fact the most general (universal) embedding of X in a compact Hausdorff space—it is the Stone-Čech compactification of X, which we will soon investigate.

References

[SS] L.A. Steen and J.A. Seebach, Jr., *Counterexamples in Topology, 2nd ed.*, Dover, 1995.