Representation Theory

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1. Representations and Characters

Let V be an n-dimensional vector space over a field F. (All vector spaces considered will be finite dimensional.) A **(linear) representation** of a group G is a homomorphism $\pi : G \to GL(V)$. Often we will take $V = F^n$, written as column vectors over a field F, so we may consider a representation to be a homomorphism $G \to GL(n, F)$.

We must first distinguish between **modular representations** (those for which F has nonzero prime characteristic) and **ordinary representations** (those in characteristic zero). Our focus will be on ordinary representation theory, which is easier and has broader applications. In this case we will usually take $F = \mathbb{C}$, which has the additional nice property of being algebraically closed. Often it is advantageous to instead take F to be a finite extension of $\mathbb{Q}$, but we will try to avoid the technicalities involved in this choice.

The **degree** of the representation π is n, the dimension of the vector space V. The representation π is **reducible** if there exists a nonzero proper subspace W (i.e. $0 < W < V$) which is invariant under *all* $\pi(g)$, $g \in G$; otherwise V is **irreducible**. A representation of degree one is called **linear**. Clearly any linear representation is irreducible, and is nothing other than a homomorphism $G \to F^\times$, where $F^\times$ is the multiplicative group of nonzero field elements. An important special case is the **trivial representation** $\pi_1(g) = \begin{pmatrix} 1 \end{pmatrix}$ of degree 1. We say that the representation π is **faithful** if $\ker \pi = 1$, or equivalently, $\pi(G) \cong G$.

To each representation $\pi : G \to GL(V)$ we associate its **character** $\chi : G \to F$ defined by $\chi(g) = \operatorname{tr} \pi(g)$, and we say that π **affords** χ. Note that $\chi(h^{-1}gh) = \operatorname{tr} \pi(h^{-1}gh) = \operatorname{tr}\big(\pi(h)^{-1}\pi(g)\pi(h)\big) = \operatorname{tr} \pi(g) = \chi(g)$, so χ is a **class function** defined on G, i.e. its value is constant on conjugacy classes. Since $\pi(1)$ is an $n \times n$ identity matrix, we have $\chi(1) = \operatorname{tr} I = n$, which is the degree of π, also called the degree of χ. Again, characters of degree one are called linear. It is easy to see that every linear character is a homomorphism, but that in general, nonlinear characters are not homomorphisms.

Two representations $\pi : G \to GL(V)$ and $\sigma : G \to GL(W)$ are **equivalent** if there exists a fixed F-isomorphism (i.e. isomorphism of vector spaces over F) $T : V \to W$ such that $T \circ \pi(g) = \sigma(g) \circ T$ for all $g \in G$. In this case we write $\pi \sim \sigma$. Note that this does indeed define an equivalence relation on the set of all representations of G over a field F. In matrix terminology, two representations $\pi, \sigma : G \to GL(n, F)$ are equivalent if there is a change of basis sugh that for all $g \in G$, there is a linear transformation represented by $\pi(g)$

with respect to the first basis, and by $\sigma(g)$ with respect to the second basis. Equivalent representations afford the same character. To see this, let $\pi \sim \sigma$ via T as above; then

$$\operatorname{tr}\pi(g) = \operatorname{tr}\bigl(T^{-1}\sigma(g)T\bigr) = \operatorname{tr}\sigma(g).$$

The converse, which is not obvious, will be proven later: if two ordinary representations afford the same character, then they are equivalent.

Let k be the number of conjugacy classes of G, and let $g_1{=}1$, g_2, ..., g_k be representatives of the distinct conjugacy classes. Recall that the conjugacy class containing g_i has size $|g_i^G| = [G : C_G(g_i)]$. The set of all class functions $G \to \mathbb{C}$ is a vector space of dimension k. We define an inner product on this space by

$$[\theta, \eta] = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\eta(g)} = \frac{1}{|G|} \sum_{i=1}^{k} |g_i^G| \theta(g_i)\overline{\eta(g_i)} = \sum_{i=1}^{k} \frac{1}{|C_G(g_i)|} \theta(g_i)\overline{\eta(g_i)}.$$

It will be shown that there are exactly k inequivalent irreducible representations $\pi_1, \ldots, \pi_k$ over $F = \mathbb{C}$, and that the corresponding characters $\chi_1, \ldots, \chi_k$ form an orthonormal basis for the space of all class functions on G.

Given two representations of G, say $\pi : G \to GL(V)$ and $\sigma : G \to GL(W)$, we may form their direct sum $\pi \oplus \sigma : G \to GL(V \oplus W)$. If we identify each linear transformation with its matrix, then

$$(\pi \oplus \sigma)(g) = \pi(g) \oplus \sigma(g) = \left(\begin{array}{c|c} \pi(g) & 0 \\ \hline 0 & \sigma(g) \end{array} \right).$$

The representation ρ is **completely reducible** if we may decompose $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_m$ where each ρ_i is irreducible; that is to say, $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ where each subspace $V_i \leq V$ is invariant under $\rho(g)$ for all $g \in G$, and moreover each of the restricted representations $\rho_i : G \to GL(V_i),\ g \mapsto \pi(g)\big|_{V_i}$ is irreducible.

Not every representation is completely reducible. For example, let G be the additive group of a field $F = GF(p)$, and consider the representation $\pi : G \to GL(2, F)$ defined by $\pi(a) = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right)$. Then $V = F^2$ is reducible but not completely reducible, since $\left\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \right\rangle$ is the unique nonzero proper invariant subspace, which therefore has no complementary invariant subspace. In the next section we will see that such anomalies cannot arise with ordinary representations: every representation $\pi : G \to GL(n, \mathbb{C})$ is completely reducible, as a direct sum of $[\chi, \chi_i]$ copies of π_i where $\{\pi_i\}$ are the irreducible complex representations of G, $\{\chi_i\}$ are the corresponding characters, and χ is the character of π.

1.1 Example. Let $G = S_3$. We will see later that G has exactly three (up to equivalence) irreducible representations over $F = \mathbb{C}$, of degree 1, 1, 2 respectively. These may be denoted

$$\pi_1(g) = \left(1 \right) \text{ for all } g \in G \text{ (the trivial representation)};$$

$$\pi_2(g) = \big(\operatorname{sgn}(g)\big) = \begin{cases} (1), & g \text{ an even permutation,} \\ (-1), & g \text{ odd;} \end{cases}$$

$\pi_3(g)$ is determined by $\pi_3\big((12)\big) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\pi_3\big((123)\big) = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \overline{\omega} \end{smallmatrix}\right)$ where $\omega \in \mathbb{C}$ is a primitive cube root of 1.

The values of the corresponding characters are conveniently expressed by the character table

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The reader should check that $\{\chi_1, \chi_2, \chi_3\}$ is an orthonormal basis for the space of class functions on G. Now consider, for example, the matrix representation $\rho : G \to GL(3, \mathbb{C})$ determined by

$$\rho\big((12)\big) = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right), \quad \rho\big((123)\big) = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix}\right).$$

The corresponding character satisfies $\chi(1) = 3$, $\chi\big((12)\big) = 1$, $\chi\big((123)\big) = 0$. One computes $[\chi, \chi_i] = 1, 0, 1$ for $i = 1, 2, 3$ respectively. Thus $\rho \sim \pi_1 \oplus \pi_3$. It is possible to decompose ρ directly with a little geometric insight: G permutes the vertices $\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ of an equilateral triangle in the plane $x + y + z = 1$, and G fixes the normal vector $v_1 = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ to this plane. Choose a new basis for $\mathbb{C}^3$ by taking v_1 together with a basis for $v_1^\perp$; say, $v_2 = \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right)$, $v_3 = \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)$. We have $V = \langle v_1 \rangle \oplus \langle v_2, v_3 \rangle$ in which both $\langle v_1 \rangle$ and $\langle v_2, v_3 \rangle$ are invariant under $\rho(G)$. Relative to the new basis $\{v_1, v_2, v_3\}$, the new matrix representation $\rho' : G \to GL(3, \mathbb{C})$ is determined by

$$\rho'\big((12)\big) = \begin{pmatrix} \rho_1'\big((12)\big) & 0 \\ 0 & \rho_2'\big((12)\big) \end{pmatrix} = \left(\begin{array}{c|cc} 1 & & \\ \hline & -1 & -1 \\ & 0 & 1 \end{array} \right),$$

$$\rho'\big((123)\big) = \begin{pmatrix} \rho_1'\big((123)\big) & 0 \\ 0 & \rho_2'\big((123)\big) \end{pmatrix} = \left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & 1 \\ & -1 & -1 \end{array} \right).$$

These matrices were found by taking images of the basis vectors; for example $\rho\big((123)\big) v_3 = v_2 - v_3$, which gives $\left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right)$ as the third column of $\rho'\big((123)\big)$. Evidently $\rho_2' \sim \pi_3$; and with a little fussing we find that $\pi_3(g) = \left(\begin{smallmatrix} \overline{\omega} & \omega \\ 1 & 1 \end{smallmatrix}\right)^{-1} \rho_2'(g) \left(\begin{smallmatrix} \overline{\omega} & \omega \\ 1 & 1 \end{smallmatrix}\right)$ for all $g \in G$.

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2. Modules

The language of modules may be considered equivalent to the language of representations. We introduce both sets of terminology since each choice of language has its merits.

Let R be a ring with identity $1 \in R$. A **(left)** R-**module** is an additive abelian group M together with a definition of scalar multiplication $rv \in M$ for $r \in R$, $v \in M$ such that

   (i) $r(v + w) = rv + rw$,
   (ii) $(r + s)v = rv + sv$,
   (iii) $r(sv) = (rs)v$,
   (iv) $1v = v$

for all $r, s \in R$, $v, w \in M$. (In case R is a field, a left R-module is the same thing as a left vector space over R.)

Recall that a vector space A over a field F which is at the same time a ring, is called an **algebra** over F, assuming some compatibility between the vector space and ring structures, namely $\lambda(xy) = x(\lambda y) = (\lambda x)y$ for all $\lambda \in F$, $x, y \in A$; also $1_F x = x$ where 1_F is the multiplicative identity in F. If A is an F-algebra, then any A-module is in particular a vector space over F, and as such we can speak of its dimension over F. We will be primarily interested in the case of the **group ring** $R = FG = \left\{ \sum_{g \in G} a_g g \ : \ a_g \in F \right\}$, which is also an algebra, called the **group algebra** of G over F. Given a representation $\pi : G \to GL(V)$, we make V into an FG-module by defining scalar multiplication as $\left(\sum_{g \in G} a_g g \right) v = \sum_{g \in G} a_g \pi(g) v \in V$; one easily checks that properties (i)–(iv) follow. Note however that this FG-module depends not only on F, G and V, but also on the choice of representation π.

The **trivial** FG-module is a one-dimensional module F which carries the trivial representation of FG. This is nothing other than F itself, where for $\sum_{g \in G} a_g g \in FG$ and $v \in F$, we have $\left(\sum_{g \in G} a_g g \right) v = \left(\sum_{g \in G} a_g \right) v \in F$.

A nonzero R-module M is **simple** if it has no nonzero proper submodules. Thus an FG-module is simple if and only if the corresponding representation of G is irreducible.

An R-module is semisimple if it is a direct sum of simple R-submodules. Thus an FG-module is semisimple if and only if the corresponding representation is completely reducible. The following is a useful criterion for semisimplicity.

2.1 Lemma. *An* R-*module* V *is semisimple if and only if for every submodule* $U \leq V$ *there exists a (complementary) submodule* U' *such that* $V = U \oplus U'$.

Proof. Suppose that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ as a direct sum of simple R-submodules, and let $U \leq V$ be any submodule. Define U' to be maximal among all submodules of V which intersect U in 0. (The class $\mathcal{C} = \left\{ \text{submodules } W \leq V \ : \ W \cap U = 0 \right\}$ contains the zero submodule and so $\mathcal{C}$ is nonempty. Since V is finite dimensional, $\mathcal{C}$ has a maximal member.) We must show that the submodule $U + U' = V$. Suppose not. Then there exists some i such that $V_i \not\subseteq U + U'$. Then $V_i \cap (U + U')$ is a proper R-submodule of V_i, and since V_i is simple, we have $V_i \cap (U + U') = 0$. Thus $(U' + V_i) \cap U = 0$, and $U' + V_i$ is an R-module properly containing U', a contradiction.

The converse follows easily by induction on $\dim V$. ◻

Let V and W be two FG-modules. Then $\mathrm{Hom}_F(V,W)$ denotes the set of all F-linear transformations $V \to W$. But $\mathrm{Hom}_R(V,W)$ denotes the set of all maps $T : V \to W$ such that $T(v+v') = T(v) + T(v')$ and $T(rv) = rT(v)$ for all $v, v' \in V$ and $r \in R$. Thus $T \in \mathrm{Hom}_R(V,W)$ if and only if
   (i) $T \in Hom_F(V,W)$, and
   (ii) T is G-equivariant, i.e. $T\big(\pi(g)v\big) = \sigma(g)T(v)$ for all $g \in G$, $v \in V$, which is to say that T 'commutes' with G in its respective actions. Here $\pi : G \to GL(V)$ and $\sigma : G \to GL(W)$ are the respective actions.
Also let $\mathrm{End}_R(V) = \mathrm{Hom}_R(V,V)$, the ring of all R-**endomorphisms** of V. In case R is an F-algebra, note that $\mathrm{Hom}_R(V,W)$ is a vector space over F and $\mathrm{End}_R(V)$ is an F-algebra. Note that $\mathrm{End}_F(V) \cong M(n,F)$ where $n = \dim V$, and $\mathrm{End}_R(V) = C_{M(n,F)}\big(\pi(G)\big) = \big\{h \in M(n,F) : h\pi(g) = \pi(g)h \text{ for all } g \in G\big\}$, the centralizer of $\pi(G)$ in $M(n,F)$.

We provide here a glossary of terminology for representations of G, together with equivalent terminology for FG-modules.

Representation-theoretic terminology	Module terminology
vector space over F with representation of G	FG-module
invariant subspace	submodule
irreducible representation	simple module
completely reducible	semisimple
equivalent representations	isomorphic modules
G-equivariant linear transformation	module homomorphism
trivial representation $g \mapsto \big(1\big)$	trivial FG-module F

Table 2.2: Glossary

The following says that if the characteristic of F is zero or a prime not dividing $|G|$, then every representation of G over F is completely reducible.

2.3 Maschke's Theorem. *If $|G|$ is not divisible by the characteristic of F, then every FG-module is semisimple.*

Proof. Let $\pi : G \to GL(V)$ be a representation where $V = F^n$, and let U be a subspace of V invariant under $\pi(G)$. By Lemma 2.1, it suffices to find an invariant subspace U' such that $V = U \oplus U'$.

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Certainly we can find a subspace W such that $V = U \oplus W$. The problem is that in general, this W is not invariant under $\pi(G)$. Let $P : V \to U$ be the projection of V onto U along W, i.e. every vector $v \in V$ is expressible uniquely as $Pv + (v - Pv)$ with $Pv \in U$, $(v - Pv) \in W$, and this property uniquely determines $P \in \operatorname{Hom}_F(V, U)$. Define $T : V \to U$ by

$$ Tv = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(h) v \,. $$

Note that $|G|$ is invertible as a field element, according to the hypothesis, so our definition of T makes sense, and it is clear that $T \in \operatorname{Hom}_F(V, U)$. We must check that moreover $T \in \operatorname{Hom}_R(V, U)$ where $R = FG$. To see this, let $g \in G$ and $v \in V$; then

$$ T\big(\pi(g)v\big) = \frac{1}{|G|} \sum_{h \in G} \pi(h^{-1}) P \pi(hg) v = \frac{1}{|G|} \sum_{u \in G} \pi(gu^{-1}) P \pi(u) v = \pi(g) T(v). $$

It is likewise easy to check that $T^2 = T$ and that $T\big|_U$ is the identity on U. Let $U' = \{v - T(v) : v \in V\}$, so that $T\big|_{U'} = 0$ and $U \cap U' = 0$. Every vector $v \in V$ is expressible as $T(v) + \big(u - T(v)\big) \in U + U'$, so $V = U \oplus U'$. Moreover U' is an FG-submodule since it is the kernel of the FG-homomorphism T. $\qquad\square$

For any ring R with identity, an important special R-module is R itself, called the **regular** R-module. Note that the submodules of this module are the left ideals of R, and the simple submodules are the minimal left ideals. We say that the ring R is semisimple if the regular module is semisimple, i.e. if $R = I_1 \oplus I_2 \oplus \cdots \oplus I_m$ as a direct sum of minimal left ideals. By Maschke's Theorem, this is true of a group ring $R = FG$ whenever $\operatorname{char}(F) \nmid |G|$. We shall see that decomposing an arbitrary R-module, depends on being able to decompose the regular R-module.

2.4 Corollary. *Suppose R is semisimple. Then every simple R-module is isomorphic to some minimal left ideal $I \subseteq R$.*

Proof. Let V be a simple R-module, and choose a nonzero $v_0 \in V$. Define $\phi : R \to V$ be $\phi(r) = rv_0$. Clearly ϕ is an R-module homomorphism; that is, $\phi \in \operatorname{Hom}_R(R, V)$. The image of ϕ is $\phi(R) = Rv_0$, a nonzero submodule of V containing v_0. Since V is simple, we have $\phi(R) = Rv_0 = V$. The kernel of ϕ is $\ker \phi = \{r \in R : rv_0 = 0\}$, a left ideal of R. Since R is semisimple, it has a left ideal J such that $R = J \oplus \ker \phi$. Thus $J \cong R/\ker \phi \cong \phi(R) = V$ as R-modules. Since V is simple, the left ideal $J \subseteq R$ is minimal. $\qquad\square$

The following shows that if $|G|$ is not divisible by $\operatorname{char} F$, then G has only finitely many irreducible representations over F (up to equivalence).

2.5 Corollary. *Suppose that R is semisimple, and choose a decomposition $R = I_1 \oplus I_2 \oplus \cdots \oplus I_m$ where each I_i is a minimal left ideal of R. Then every simple R-module is isomorphic to I_i for some i, $1 \leq i \leq m$. In particular, R has only finitely many simple modules (up to isomorphism).*

Proof. Let M be a simple R-module. By Corollary 2.4, we may assume that M is a minimal left ideal of R. Since R is semisimple, it has a left ideal J such that $R = M \oplus J$. Now J is a proper ideal, so there exists i such that $I_i \not\subseteq J$. Then $I_i \cap J$ is a proper R-submodule of I_i, and since I_i is simple, we have $I_i \cap J = 0$. Now

$$I_i \cong I_i/(I_i \cap J) \cong (I_i + J)/J \leq R/J \cong M.$$

Since M is simple, we in fact have $M \cong I_i$. $\qquad\square$

3. Schur's Lemma

A number of related results, beginning with the following, are collectively known as Schur's Lemma. First, suppose that $\pi : G \to GL(n, F)$ and $\sigma : G \to GL(m, F)$ are two irreducible representations. Our first result, Lemma 3.1, says that if T is an $m \times n$ matrix over F such that $T\pi(g) = \sigma(g)T$ for all $g \in G$, then either $T = 0$ or T is square invertible and $\pi \sim \sigma$. We have phrased the statement and proof more concisely, however, using module terminology.

Recall that a ring A with identity 1 such that the nonzero elements are invertible, is called a **division ring** or **skewfield**. A field is the same thing as a commutative division ring.

3.1 Lemma. *Let M and N be simple R-modules. If M and N are not isomorphic, then $Hom_R(M, N) = 0$. If M and N are isomorphic, then $Hom_R(M, N) \cong \mathrm{Hom}_R(M, M) = \mathrm{End}_R(M)$ is a division ring.*

Proof. Suppose that $\phi : M \to N$ is a homomorphism of simple R-modules. Then $\ker \phi \leq M$ and $\phi(M) \leq N$ are R-submodules. Since M and N are simple, either
  (i) $\ker \phi = M$, $\phi(M) \cong M/\ker \phi \cong M/M = 0$, so $\phi = 0$, or
  (ii) $\ker \phi = 0$, $\phi(M) \cong M/0 \cong M$ and ϕ is an R-isomorphism.
Clearly $\mathrm{End}_R(M)$ is a ring with identity. If ϕ is a nonzero R-endomorphism of M, then the above shows that $\phi : M \to M$ is an isomorphism; in this case there is an inverse map $\phi^{-1} : M \to M$, and it is easy to see that $\phi^{-1} \in \mathrm{End}_R(M)$. $\qquad\square$

The following corollary says that if $\pi : G \to GL(n, \mathbb{C})$ is irreducible, and T is any $n \times n$ complex matrix commuting with all matrices $\pi(G)$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$. In other words, the centralizer of $\pi(G)$ in $M(n, \mathbb{C})$ consists of scalar multiples of the identity. This is true more generally for representations over any algebraically closed field, regardless of the characteristic.

3.2 Corollary. *Let F be an algebraically closed field, G a finite group, $R = FG$ the group algebra, and M a finite dimensional simple R-module. Then $End_R(M) = \{\lambda I : \lambda \in F\} \cong F$.*

Proof of Corollary 3.2. Let $\phi : M \to M$ be an R-module homomorphism. Then in particular ϕ is an F-linear transformation. Since F is algebraically closed, we may choose $\lambda \in F$ to be a root of the characteristic polynomial of ϕ, and there exists a nonzero $v \in M$ such that $\phi(v) = \lambda v$. Define $U = \ker(\phi - \lambda I)$, the eigenspace with eigenvalue λ. Then for any $u \in U$ and $r \in R$, we have $\phi(ru) = r\phi(u) = r\lambda u = \lambda(ru)$, so U is an R-submodule of M. Since $U \neq 0$ and M is simple, we have $U = M$, which says that $\phi(u) = \lambda u$ for all $u \in M$. $\qquad\square$

To see why the hypothesis on F was necessary, observe that there is a representation of $G = \{1, g, g^2, g^3\} \cong C_4$ of degree 2 determined by $\pi(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. If we take $F = \mathbb{R}$ then π is irreducible and the centralizer of $\pi(G)$ in $M(2, \mathbb{R})$ is $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$, which consists of more than just scalar multiples of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If instead we take $F = \mathbb{C}$, then π is not irreducible; we have $\mathbb{C}^2 = \left\langle \begin{pmatrix} 1 \\ i \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\rangle$ as a direct sum of one-dimensional submodules.

4. The Regular Module

Let R be a semisimple ring (for example a group ring FG such that $\mathrm{char}(F) \nmid |G|$). Our aim is to be able to decompose every R-modules as a direct sum of irreducibles. We will accomplish this by decomposing, in particular, the regular R-module R. We begin with two examples.

4.1 Example. We continue with the notation of Example 1.1 for $G = S_3$. The group ring $R = \mathbb{C}G$ is six-dimensional. We know R is semisimple by Maschke's Theorem, but here we provide an explicit decomposition of R into minimal left ideals. Consider the following elements of R:

$$v_1 = \sum_{g \in G} g = (1) + (123) + (132) + (12) + (23) + (13),$$
$$v_2 = (1) + (123) + (132) - (12) - (23) - (13),$$
$$v_3 = (12) + \omega(13) + \overline{\omega}(23),$$
$$v_4 = (1) + \omega(123) + \overline{\omega}(132),$$
$$v_5 = (1) + \overline{\omega}(123) + \omega(132),$$
$$v_6 = (12) + \overline{\omega}(13) + \omega(23).$$

Then $R = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \langle v_3, v_4 \rangle \oplus \langle v_5, v_6 \rangle$ where each of the four summands is a minimal left ideal of R. The representation of G on each of these ideals is given by π_1, π_2, π_3 and π_3 respectively. Thus the irreducible representations occur with multiplicities 1, 1 and 2.

4.2 Example. Let $R = M(n, F)$, the ring of all $n \times n$ matrices over an arbitrary field F. For $1 \leq i \leq n$, let

$$I_i = \begin{pmatrix} 0 & \cdots & * & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & * & \cdots & 0 \end{pmatrix},$$

the set of all matrices in R with arbitrary entries in the i-th column and zeroes elsewhere. Then $R = I_1 \oplus I_2 \oplus \cdots \oplus I_n$ where each I_i is a minimal left ideal. Thus R is semisimple.

Our goal is to express every group ring $\mathbb{C}G$ as a direct sum of simple ideals isomorphic to $M(n_i, \mathbb{C})$. An example of this, using the notation of Example 4.1 for $G = S_3$, is

$$\mathbb{C}G = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \langle v_3, v_4, v_5, v_6 \rangle \cong M(1, \mathbb{C}) \oplus M(1, \mathbb{C}) \oplus M(2, \mathbb{C}).$$

In this case $\langle v_3, v_4, v_5, v_6 \rangle = \langle v_3, v_4 \rangle \oplus \langle v_5, v_6 \rangle$ as a direct sum of minimal left ideals. There are lots of additional submodules of R isomorphic to $\langle v_3, v_4 \rangle$, all of the form $\langle \alpha v_3 + \beta v_5, \ \alpha v_4 + \beta v_6 \rangle$ for constants $\alpha, \beta \in \mathbb{C}$, all of which are submodules of $\langle v_3, v_4, v_5, v_6 \rangle$. Therefore we may obtain $\langle v_3, v_4, v_5, v_6 \rangle$ as the (not direct!) sum of all ideals of R which are isomorphic (as R-modules) to $\langle v_3, v_4 \rangle$.

Let us express this idea more generally. Let R be a semisimple ring, and let $M_1, M_2, \ldots, M_k$ be the distinct irreducible R-modules. (We saw in Corollary 2.5 that there are only finitely many irreducible R-modules up to isomorphism.) Let $M_i(R)$ be the sum (not direct!) of all minimal left ideals of R isomorphic to M_i (as R-modules), called the M_i-**homogeneous part** of R.

Returning to Example 4.2 with $R = M(n, F)$, we have $V = F^n$, which is a simple R-module under the usual matrix action. Note that every $I_i \cong V$ as R-modules. It follows from Corollary 2.5 that V is the *unique* simple R-module (up to isomorphism). In this case R has only one V-homogeneous part, $V(R) = I_1 + I_2 + \cdots + I_n + (\text{other such submodules}) = R$. Here again we get our wish: the V-homogeneous part is $R = M(n, F)$, a full matrix algebra.

Before we prove our desired decomposition theorem, we observe the following easy result.

4.3 Lemma. *The ring $End_R(R)$ is anti-isomorphic to R.*

Proof. For each $a \in R$, let $\rho_a : R \to R$ denote right-multiplication by a; that is, for $x \in R$ we have $\rho_a(x) = xa \in R$. Now $\rho_a \in End_R(R)$, since for $r, s, x, y \in R$ we have $\rho_a(rx + sy) = (rx + sy)a = r(xa) + s(ya) = r\rho_a(x) + s\rho_a(y)$. The map $\rho : R \to End_R(R)$ given by $a \to \rho_a$ is one-to-one since if $\rho_a = \rho_b$, then $\rho_a(1) = \rho_b(1)$ yields $a = b$. Also ρ is surjective, since if $\psi \in End_R(R)$, we may let $a = \psi(1)$; then $\psi(x) = \psi(x1) = x\psi(1) = xa = \rho_a(x)$ for all $x \in R$ so that $\rho_a = \psi$. Also ρ is an anti-isomorphism since $\rho_{ab}(x) = xab = \rho_b(\rho_a(x))$. $\quad\square$

A number of results are collectively known as Wedderburn's Theorem. The spirit of these is that every semisimple algebra is a direct sum of simple algebras, each of which is a full matrix algebra over a division ring extending the original field. Parts (i) and (ii) of the following are part of Wedderburn's Theorem.

4.4 Theorem. *Let $M_1, M_2, \ldots, M_k$ be the distinct irreducible R-modules (up to isomorphism) for the group ring $R = \mathbb{C}G$.*

(i) *Each homogeneous part $M_i(R)$ is a simple ideal of R. We have a ring isomorphism $M_i(R) \cong M(n_i, \mathbb{C})$ where n_i is the number of minimal left ideals (each isomorphic to M_i) in a direct sum decomposition of $M_i(R)$. Also $n_i = \dim M_i$.*

(ii) *$R = M_1(R) \oplus M_2(R) \oplus \cdots \oplus M_k(R) \cong M(n_1, \mathbb{C}) \oplus M(n_2, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$ as rings. In particular, $|G| = n_1^2 + n_2^2 + \cdots + n_k^2$.*

(iii) *The number k equals the number of conjugacy classes of G.*

Proof. Decompose $R = I_1 \oplus I_2 \oplus \cdots \oplus I_m$ as a direct sum of minimal left ideals. With respect to this decomposition, every R-endomorphism φ of R has a unique matrix representation

$$
\begin{pmatrix}
\varphi_{11} & \cdots & \varphi_{1m} \\
\vdots & \ddots & \vdots \\
\varphi_{m1} & \cdots & \varphi_{mm}
\end{pmatrix}
$$

where $\varphi_{ij} \in \mathrm{Hom}_R(I_i, I_j)$. (This means that for $x \in R$, we express $x = \sum_j x_j$ uniquely where $x_j \in I_j$, and then $\phi(x) = \sum_i \left(\sum_j \varphi_{ij}(x_j) \right)$ where $\sum_j \varphi_{ij}(x_j)$ is the component of $\varphi(x)$ in I_i.)

Recall from Lemma 3.1 and Corollary 3.2 that $\mathrm{Hom}_R(I_i, I_j) = 0$ if $I_i \not\cong I_j$ as R-modules; otherwise $\mathrm{Hom}_R(I_i, I_j) \cong \mathbb{C}$. We may assume that $M_1(R) = I_1 \oplus \cdots \oplus I_{n_1}$, $M_2(R) = I_{n_1+1} \oplus \cdots \oplus I_{n_1+n_2}$, $\ldots$, $M_k(R) = I_{m-n_k+1} \oplus \cdots \oplus I_m$. It follows that $\mathrm{End}_R(R) \cong M(n_1, \mathbb{C}) \oplus M(n_2, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$. The previous Lemma gives an anti-automorphism from $R \to \mathrm{End}_R(R)$; also the transpose map gives an anti-automorphism $M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C}) \to M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$. Composing maps in the right order gives an isomorphism $R \to M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$. This isomorphism takes $M_i(R)$ to the i-th summand $M(n_i, \mathbb{C})$, which is clearly a simple ideal of R (cf. Example 4.2). Also the minimal left ideals of $M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k, \mathbb{C})$ contained in $M(n_i, \mathbb{C})$ are n_i-dimensional, so $\dim I_j = n_i$ for every summand $I_j \cong M_i$. Comparing dimensions gives $|G| = n_1^2 + n_2^2 + \cdots + n_k^2$. This proves (i) and (ii).

To prove (iii), first note that each $M(n_i, \mathbb{C})$ has a one-dimensional center consisting of scalar transformations; therefore $Z\big(M(n_1, \mathbb{C}) \oplus \cdots \oplus M(n_k \mathbb{C})\big)$ is k-dimensional, a direct sum of scalar transformations in the k summands. But $\sum_{g \in G} a_g g \in Z(R)$ if and only if the coefficients a_g are constant on conjugacy classes of G; this is obvious from comparing coefficients of $g \in G$ in $\sum_{g \in G} a_g g = h\big(\sum_{g \in G} a_g g\big)h^{-1} = \sum_{g \in G} a_{h^{-1}gh} g$ for $h \in G$. Thus if $K_1, K_2, \ldots, K_\ell$ are the conjugacy classes of G, then the elements $\sum_{g \in K_i} g$ for $1 \leq i \leq \ell$ form a basis of $Z(R)$ and we must have $\ell = k$. $\square$

5. Orthogonality Relations

Let K_1, K_2, ..., K_k be the conjugacy classes of G. We have seen that G has exactly k inequivalent complex representations π_1, π_2, ..., π_k. Let χ_1, χ_2, ..., χ_k be the corresponding characters, called the **irreducible characters** of G. Our goal in this section is to prove that $\mathrm{Irr}_G \overset{\mathrm{def}}{=} \{\chi_1, \ldots, \chi_k\}$ is an orthonormal basis for the space of class functions on G. First, we prove a lemma. *Note:* All representations and characters in this section are over the field $\mathbb{C}$.

5.1 Lemma. *Let χ be any character of G. Then $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.*

Proof. Let $n = |G|$, so that $g^n = 1$. If π is a representation of G of degree r which affords χ, then $\pi(g)^n = \pi(g^n) = I$. Therefore every eigenvalue of $\pi(g)$ is an n-th root of unity. Let λ_1, λ_2, ..., λ_r be the (not necessarily distinct) eigenvalues of $\pi(g)$. Then $\chi(g^{-1}) = \sum_i \lambda_i^{-1} = \sum_i \overline{\lambda_i} = \overline{\chi(g)}$. $\square$

5.2 Lemma. *Let $\pi : G \to GL(r, \mathbb{C})$ and $\sigma : G \to GL(s, \mathbb{C})$ be inequivalent irreducible matrix rerpesentations of G. Let $\pi_{ij}(g)$ be the (i,j)-entry of $\pi(g)$ $(1 \le i, j \le r)$, and similarly for σ. For all i, j, k, ℓ we have*

(i) $\sum\limits_{g \in G} \pi_{ij}(g)\sigma_{k\ell}(g^{-1}) = 0$, *and*

(ii) $\sum\limits_{g \in G} \pi_{ij}(g)\pi_{k\ell}(g^{-1}) = \dfrac{|G|}{r}\delta_{i\ell}\delta_{jk}.$

Proof. (i) Let $V = \mathbb{C}^r$ and $W = \mathbb{C}^s$ be the modules on which G acts via π and σ respectively. Let T be the $s \times r$ matrix whose (j, k)-entry is 1, and all other entries are zero. We define a linear transformation $\varphi = \varphi_T \in \mathrm{Hom}_C(V, W)$ by

$$\varphi = \sum_{g \in G} \sigma(g)T\pi(g^{-1}).$$

If $h \in G$ then

$$\varphi\pi(h) = \sum_{g \in G} \sigma(g)T\pi(g^{-1}h) = \sum_{x \in G} \sigma(hx)T\pi(x^{-1}) = \sigma(h)\varphi.$$

That is, $\varphi \in \mathrm{Hom}_{CG}(V, W)$. By Schur's Lemma 3.1, we have $\varphi = 0$. The (i, ℓ)-entry of φ is $0 = \sum_{g \in G} \pi_{ij}(g)\sigma_{k\ell}(g^{-1})$.

(ii) As above, we obtain $\varphi = \sum_{g \in G} \pi_{ij}(g)T\pi_{k\ell}(g^{-1}) \in \mathrm{End}_{\mathbb{C}G}(V)$. By Schur's Lemma 3.2, we have $\varphi = \lambda_{jk}I$. Comparing (i, ℓ)-entries on both sides of the latter equality yields

(5.3) $\qquad \sum\limits_{g \in G} \pi_{ij}(g)\pi_{k\ell}(g^{-1}) = \lambda_{jk}\delta_{i\ell}.$

Interchanging $i \leftrightarrow \ell$ and $j \leftrightarrow k$ in (5.3) gives $\sum_{g \in G} \pi_{\ell k}(g)\pi_{ji}(g^{-1}) = \lambda_{i\ell}\delta_{jk}$; however, replacing $h = g^{-1}$ in the latter summation gives an expression identical to (5.3). Therefore the expression in (5.3) becomes $\lambda_{jk}\delta_{i\ell} = \lambda_{i\ell}\delta_{jk}$, which simplifies to $\lambda\delta_{i\ell}\delta_{jk}$ where $\lambda = \lambda_{ii}$, independent of i. Substituting this into (5.3) yields

(5.4) $\sum_{g\in G} \pi_{ij}(g)\pi_{k\ell}(g^{-1}) = \lambda\delta_{i\ell}\delta_{jk}$.

In particular, $\lambda = \sum_{g\in G} \pi_{ij}(g)\pi_{ji}(g^{-1})$. Summing the latter expression over j yields

$$r\lambda = \sum_{j=1}^{r}\Big(\sum_{g\in G} \pi_{ij}(g)\pi_{ji}(g^{-1})\Big) = \sum_{g\in G}\Big(\sum_{j=1}^{r} \pi_{ij}(g)\pi_{ji}(g^{-1})\Big).$$

But $\sum_{j} \pi_{ij}(g)\pi_{ji}(g^{-1})$ is just the (i,i)-entry of $\pi(g)\pi(g^{-1}) = I$, which is 1. Thus $r\lambda = \sum_{g\in G} 1 = |G|$, i.e. $\lambda = |G|/r$. Substituting this into (5.4) gives (ii). $\qquad\square$

Recall that a class function on G is a function $G \to \mathbb{C}$ which is constant on each conjugacy class K_i. Such functions form a k-dimensional vector space over $\mathbb{C}$, which is an inner product space where for any two class functions θ, η we define

$$[\theta, \eta] = \frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\eta(g)} = \frac{1}{|G|}\sum_{i=1}^{k}|K_i|\,\theta(g_i)\overline{\eta(g_i)} = \sum_{i=1}^{k}\frac{1}{|C_G(g_i)|}\theta(g_i)\overline{\eta(g_i)}$$

where $g_i \in K_i$ are representatives of the conjugacy classes.

5.5 Theorem (Frobenius). $Irr_G = \{\chi_1, \ldots, \chi_k\}$ *is an orthonormal basis for the space of complex-valued class functions on G.*

Proof. Let χ and ψ be irreducible characters of G afforded by representations π and σ of degree r and s respectively. Then in the notation of Lemma 5.2, we have

$$|G|[\chi, \psi] = \sum_{g\in G}\chi(g)\overline{\psi(g)} = \sum_{g\in G}\chi(g)\psi(g^{-1}) = \sum_{g\in G}\sum_{i=1}^{r}\sum_{j=1}^{s}\pi_{ii}\sigma_{jj}(g^{-1})$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g\in G}\pi_{ii}\sigma_{jj}(g^{-1}) = 0\,,$$

$$|G|[\chi, \chi] = \sum_{g\in G}\chi(g)\overline{\chi(g)} = \sum_{g\in G}\chi(g)\chi(g^{-1}) = \sum_{g\in G}\sum_{i=1}^{r}\sum_{j=1}^{s}\pi_{ii}\pi_{jj}(g^{-1})$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{g\in G}\pi_{ii}\pi_{jj}(g^{-1}) = \sum_{i=1}^{r}\sum_{j=1}^{s}\frac{|G|}{r}\delta_{ij}\delta_{ij} = r\frac{|G|}{r} = |G|\,. \qquad\square$$

Now let $\pi : G \to GL(n, \mathbb{C})$ be any representation, and let χ be its character. By Maschke's Theorem, π is equivalent to a direct sum of copies of the irreducible representations $\pi_1, \ldots, \pi_k$ of G, with corresponding multiplicities $n_1, \ldots, n_k$. By the previous theorem the multiplicities are determined by $n_i = [\chi, \chi_i]$. Consequently, we have

5.6 Corollary. *Any complex representation of a given group G is determined (to within equivalence) by its character.*

If G is a finite group, a **character table** for G is a $k \times k$ matrix with rows indexed by the irreducible complex characters χ_i of G, and columns indexed by the conjugacy classes K_j of G, having (i,j)-entry $\chi_i(g_j)$ where $g_j \in K_j$. Usually we order our indices such that χ_1 is the trivial character and $K_1 = \{1\}$; then the first row of the character table consists of 1's, and the first column gives the degrees $n_i = \chi_i(1)$ of the irreducible representations of G.

Let M be the $k \times k$ character table of G (as above), and let $D = \operatorname{diag}(d_1, d_2, \ldots, d_k)$ where $d_i = |C_G(g_i)|^{-1/2}$. Then by Theorem 5.5, the rows of MD form an orthonormal basis of $\mathbb{C}^k$ with respect to the standard inner product, i.e. MD is unitary in the usual sense, i.e. $(MD)\overline{(MD)}^{\mathrm{T}} = I$. Therefore the columns of MD also form an orthonormal basis of $\mathbb{C}^k$, which proves the following.

5.7 Corollary. *If $\mathrm{Irr}_G = \{\chi_1, \ldots, \chi_k\}$, then $\sum\limits_{i=1}^{k} \chi_i(g_j)\overline{\chi_i(g_k)} = |C_G(g_j)|\delta_{jk}$.*

Before continuing, we show the following, which will be useful later.

5.8 Theorem. *The set of characters of G is closed under addition and under multiplication.*

Proof. Let $\pi : G \to GL(r, \mathbb{C})$ and $\sigma : G \to GL(s, \mathbb{C})$ be representations, with associated characters χ, η. Then $\pi \oplus \sigma$ is a representation of degree $r + s$ which affords $\chi + \eta$, so the set of characters of G is closed under addition.

Also $\pi \otimes \sigma$ is a representation of degree rs, where $(\pi \otimes \sigma)(g) = \pi(g) \otimes \sigma(g)$. Note that

$$
\begin{aligned}
(\pi \otimes \sigma)(gh) = \pi(gh) \otimes \sigma(gh) &= \big(\pi(g) \otimes \sigma(g)\big)\big(\pi(g) \otimes \sigma(g)\big) \\
&= (\pi \otimes \sigma)(g)(\pi \otimes \sigma)(g),
\end{aligned}
$$

so $\pi \otimes \sigma : G \to GL(rs, \mathbb{C})$ is a homomorphism. Also $\operatorname{tr}(\pi(g) \otimes \sigma(g)) = (\operatorname{tr}\pi(g))(\operatorname{tr}\sigma(g)) = \chi(g)\eta(g)$, which proves that the product of two characters is a character. $\qquad\square$

6. Linear Characters

An important special case, historically the first case to be considered, is the case G is abelian. In this case, $G = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_\ell \rangle \cong C_{m_1} \times C_{m_2} \times \cdots \times C_{m_\ell}$, $m = |G| = m_1 m_2 \cdots m_\ell$. We construct m linear characters of G as follows:

$$\chi_{\mathbf{f}}\left(x_1^{e_1} x_2^{e_2} \cdots x_\ell^{e_\ell}\right) = \exp\left(2\pi i \sum_{i=1}^{\ell} \frac{f_i e_i}{m_i}\right), \quad \mathbf{f} = (f_1, \ldots, f_\ell), \quad 0 \le f_i < m_i.$$

Since the $\chi_{\mathbf{f}}$'s are distinct linear characters, they correspond to m inequivalent irreducible characters of degree 1. But G has exactly m conjugacy classes, each class a singleton $K_i = \{g_i\}$. So by Theorem 4.4, these are *all* the irreducible characters of G. Indeed, we shall see that the abelian property of G is equivalent to the property that every irreducible character of G is linear. More generally, we wish to find all linear characters of a given finite group G.

We first recall some notation and terminology. We define the **commutator** of two elements $x, y \in G$ by

$$[x, y] = x^{-1} y^{-1} x y \in G.$$

We define the **commutator subgroup** of two given subgroups $H, K \le G$ by

$$[H, K] = \big\langle [h, k] : h \in H, \ k \in K \big\rangle.$$

In particular the **derived subgroup** of G is $G' = [G, G]$. Note that subgroups H and K commute if and only if $[H, K] = 1$. Also G is abelian if and only if $G' = 1$; in the opposite extreme case that $G' = G$, we say that the group G is **perfect**. For example every nonabelian simple group is perfect; and if $n \ge 2$, then $SL(n, F)$ is perfect except when $n = 2$ and $q \le 3$.

6.1 Lemma. *Let G be a finite group. Then $G' \trianglelefteq G$, and the quotient group G/G' is abelian. Moreover, G' is the unique smallest normal subgroup of G whose quotient group is abelian.*

Proof. For $x, y, g \in G$, we have $[x, y]^g = [x^g, y^g]$. Therefore the set of all commutators in G is invariant under conjugation, so the commutators form a normal subgroup. For $xG', yG' \in G/G'$, we have $(xG')(yG') = (yG')(xG')[x, y] = (yG')(xG')$ since $[x, y] \in G'$. Thus G/G' is abelian.

Conversely, suppose that $K \trianglelefteq G$ such that G/N is abelian. For all $x, y \in G$, we have

$$N = [xN, yN] = (xN)^{-1}(yN)^{-1}(xN)(yN) = (x^{-1}y^{-1}xy)N = [x, y]N,$$

so $[x, y] \in N$ and $G' \le N$. $\qquad\square$

Now let G be any finite group, and let $\gamma : G \to G/G'$ be the canonical projection $x \mapsto xG'$. If χ is any irreducible character of G/G' then χ is linear, so the composition

$$G \xrightarrow{\ \gamma\ } G/G' \xrightarrow{\ \chi\ } \mathbb{C}^\times$$

is a homomorphism, and hence is a linear character of G.

Conversely, suppose $\eta : G \to \mathbb{C}^\times$ is a linear character. Then $\eta(G)$ is a finite subgroup of $\mathbb{C}^\times$, which is abelian. So $G/\ker(\eta) \cong \eta(G)$ is abelian. By Lemma 6.1 we have $\ker(\eta) \supseteq G'$. Therefore there exists a homomorphism $\chi : G/G' \to \mathbb{C}^\times$ such that $\eta = \chi \circ \gamma$. This proves the following.

6.2 Corollary. *The number of linear characters of G equals $[G : G']$. These linear characters are just the functions $\chi \circ \gamma$ such that $\chi \in \operatorname{Irr}_{G/G'}$ where $\gamma : G \to G/G'$ is the canonical projection.*

6.3 Example. Let $G = A_4$. Then G has four conjugacy classes, with representatives (1), $(12)(34)$, (123) and (124); hence G has 4 irreducible characters. Since $G' = \langle (12)(34), (13)(24) \rangle \cong 2^2$ of index 3, G has exactly three linear characters χ_1, χ_2, χ_3 determined by the three homomorphisms $G/G' \to \langle \omega \rangle$ where $\omega \in \mathbb{C}$ is a primitive cube root of 1. Solving $|G| = 12 = 1^2 + 1^2 + 1^2 + n_4^2$ gives $\chi_4(1) = n_4 = 3$. The orthogonality relations completely determine the character table of G to be the following:

$|C_G(g)|$	12	4	3	3
g	(1)	$(12)(34)$	(123)	(124)
χ_1	1	1	1	1
χ_2	1	1	ω	$\overline{\omega}$
χ_3	1	1	$\overline{\omega}$	ω
χ_4	3	-1	0	0

This doesn't indicate how to find a representation of degree 3 which affords χ_4, but this will come in the next section.

7. Permutation Modules

It is easy to turn any permutation action of a group, into a matrix representation: simply represent each of the permutations by its corresponding matrix. We may use ideas from permutation groups to aid in understanding linear representations; and conversely, some representation theory is very useful in studying permutation groups.

So we begin with a permutation action of a finite group G on a finite set Ω of cardinality n. By definition, such an action is a homomorphism $\pi : G \to \operatorname{Sym} \Omega$, where $\operatorname{Sym} \Omega$ is the group of all bijections $\Omega \to \Omega$. Often we will simply take $\Omega = \{1, 2, \ldots, n\}$, so that $\operatorname{Sym} \Omega = S_n$. Now choose a field F, and let V be the n-dimensional vector space with basis Ω. (If using Ω as a basis causes notational confusion, we instead introduce n new symbols e_X for $X \in \Omega$, and use these symbols as our basis.) There is a unique extension of π to a linear representation on V, also denoted π; namely, $\pi : G \to GL(V)$ is determined by $\left(\sum_{X \in \Omega} a_X X \right)^{\pi(g)} = \sum_{X \in \Omega} a_X X^{\pi(g)}$. If no confusion is possible, we write X^g in place of $X^{\pi(g)}$, and v^g in place of $v^{\pi(g)}$ for $v \in V$. Note that in this description, linear transformations act on the right; this is done to conform with notation from permutation groups, and it causes no significant problems with our linear representation theory, except that we must now represent a typical vector $\sum_{X \in \Omega} a_X X \in V$ by a *row* vector $\left(a_X : X \in \Omega \right)$. If we identify each permutation $\pi(g)$ (and the corresponding linear transformation) with its matrix with respect to the basis Ω, we have $\pi(g) = \left(\delta_{X^g, Y} : X, Y \in \Omega \right)$, a permutation matrix. We consider V as a module for FG, the **permutation module**. Note that

$n = |\Omega| = \dim V$ is the degree of π in either sense: as a permutation representation, or as a linear representation.

The character χ associated to π is called the **permutation character**. Note that $\chi(g) = \operatorname{tr} \pi(g) = \sum_{X \in \Omega} \delta_{X^g, X} = \left| \{ X \in \Omega \ : \ X^g = X \} \right|$. In other words, $\chi(g)$ is just the number of points of Ω fixed by g.

An example is the representation $\rho : S_3 \to GL(3, \mathbb{C})$ of Example 1.1, in which the associated space $V = \mathbb{C}^3$ is the permutation module for $G = S_3$ in its natural representation of degree 3 on $\Omega = \{1, 2, 3\}$. In this case the natural basis for V is $\{ e_X \ : \ X \in \Omega \} = \{ e_1, e_2, e_3 \}$, the standard basis for $\mathbb{C}^3$.

In the following, we denote the principal character by $\chi_1 = 1$ as usual.

7.1 Proposition. *The number of orbits of G on Ω is $[\chi, \chi_1]$.*

Proof. Note that $[\chi, \chi_1] = \frac{1}{|G|} \sum_{g \in G} \chi(g)$ is the average number of fixed points of the permutations $\pi(G)$.

Consider first the case that G acts transitively on Ω. Let $S = \{ (X, g) : X \in \Omega, \ g \in G, \ X^g = X \}$. We count $|S|$ in two different ways. For each $g \in G$, the number of pairs (X, g) such that $X^g = X$ is $\chi(g)$, so $|S| = \sum_{g \in G} \chi(g) = |G| [\chi, \chi_1]$. On the other hand, for each point $X \in \Omega$, the number of pairs (X, g) such that $X^g = X$, is the order of the stabilizer G_X, so $|S| = |\Omega| |G_X| = |G|$. This gives $[\chi, \chi_1] = 1$ as required.

For the general case, let $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_w$ as a disjoint union of orbits for G. Clearly this gives a decomposition $V = \langle \Omega \rangle = \langle \Omega_1 \rangle \oplus \langle \Omega_2 \rangle \oplus \cdots \oplus \langle \Omega_w \rangle$ where each $\langle \Omega_i \rangle$ is invariant under $\pi(G)$. Let $\pi_{\Omega_i}(g)$ be the restriction of $\pi(g)$ to $\langle \Omega_i \rangle$, and let χ_{Ω_i} be the character afforded by π_{Ω_i}. Then $\pi = \pi_{\Omega_1} \oplus \cdots \oplus \pi_{\Omega_w}$ and $\chi = \sum_{i=1}^{w} \chi_{\Omega_i}$. Since G acts transitively on each orbit, by the previous case we have $[\chi_{\Omega_i}, \chi_1] = 1$, and so $[\chi, \chi_1] = \sum_{i=1}^{w} [\chi_{\Omega_i}, \chi_1] = w$, the number of orbits. $\qquad \square$

It is evident from the latter proof that if G is not transitive on Ω, then the permutation module decomposes as a direct sum of submodules $\langle \Omega_i \rangle$. If we understand the representations π_{Ω_i}, where G acts transitively on each orbit Ω_i, then by taking direct sums we can expect to understand π. So for the rest of this Section we will assume that G acts transitively on Ω. Also we will take $F = \mathbb{C}$. Please keep in mind Example 1.1 as you read the following statements.

For a transitive action of G on Ω, Proposition 7.1 shows that the trivial representation occurs with multiplicity 1 in π. Indeed, it is easy to see that if $v_1 = \sum_{X \in \Omega} X$, represented by the all-1's vector, then $\langle v_1 \rangle$ is a one-dimensional submodule of V, and in fact this is the trivial module bearing the trivial representation $\pi_1(g) = \begin{pmatrix} 1 \end{pmatrix}$. Over the complex numbers, π is completely reducible, so $V = \langle v_1 \rangle \oplus U$ for some $(n-1)$-dimensional submodule U. It is easy to see that $U = \{ \sum_{X \in \Omega} a_X X \ : \ \sum a_X = 0 \}$. (The fact that $\langle v_1 \rangle \cap U = 0$ follows from the fact that $\mathbb{C}$ has characteristic 0.) This gives a decomposition $\pi \sim \pi_1 \oplus \pi'$, where $\pi'(g)$ is the restriction of $\pi(g)$ to U. Let χ' be the character of degree $n-1$ afforded by π'. Then $[\chi', \chi_1] = [\chi, \chi_1] - [\chi_1, \chi_1] = 1 - 1 = 0$, so the trivial representation does not occur in π'. Is π' irreducible? The answer is provided by the following.

7.2 Theorem. *Suppose that G acts transitively on Ω, and let χ be the permutation character. Then $[\chi, \chi]$ is the number of orbits of G_X, the point stabilizer, on Ω. In particular, G acts doubly transitively if and only if $[\chi, \chi] = 2$, if and only if π' (as above) is irreducible.*

Proof. Note that G acts on $\Omega \times \Omega$ componentwise via $(Y, Z)^g = (Y^g, Z^g)$. Since G acts transitively on Ω, the number of point orbits of G_X on Ω equals the number of orbits of G on $\Omega \times \Omega$. Let $V = \langle \Omega \rangle$ be the permutation module for the representation π of G, so that $\dim V = n = |\Omega|$. Then the n^2-dimensional permutation module for the action of G on $\Omega \times \Omega$ is $V \otimes V \cong \langle \Omega \times \Omega \rangle$ (cf. Theorem 5.8). To see this, we have an obvious $\mathbb{C}$-linear transformation $\varphi : V \otimes V \to \langle \Omega \times \Omega \rangle$ defined by $\sum_{Y,Z \in \Omega} a_{Y,Z} Y \otimes Z \mapsto \sum_{Y,Z \in \Omega} a_{Y,Z} (Y, Z)$. Furthermore, φ is an isomorphism of $\mathbb{C}G$-modules, since

$$\varphi\Big(\big(\sum_{Y,Z} a_{Y,Z} Y \otimes Z\big)^g\Big) = \varphi\Big(\sum_{Y,Z} a_{Y,Z} Y^g \otimes Z^g\Big) = \sum_{Y,Z} a_{Y,Z} (Y^g, Z^g) = \Big(\sum_{Y,Z} a_{Y,Z} (Y, Z)\Big)^g.$$

By Theorem 5.8, the permutation character for G on $\Omega \times \Omega$ is χ^2. Therefore the number of orbits of G_X on Ω, equals the number of orbits of G on $\Omega \times \Omega$, which by Proposition 7.1 is $[\chi^2, \chi_1] = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2 = [\chi, \chi]$.

Now G acts doubly transitively on Ω if and only if G_X has only two orbits $\{X\}$ and $\Omega \smallsetminus \{X\}$, which by the above, is equivalent to $[\chi, \chi] = 2$. Let k be the number of conjugacy classes of G, and let $\pi_1, \pi_2, \ldots, \pi_k$ be the irreducible complex representations of G, with $\mathrm{Irr}_G = \{\chi_1, \chi_2, \ldots, \chi_k\}$. In the preceding notation, we have $\pi = \pi_1 \oplus \pi'$ where π' decomposes as a direct sum of irreducibles, in which the irreducible π_i occurs with multiplicity r_i, say. Then $\chi = \chi_1 + \sum_{i=2}^{k} r_i \chi_i$, so $[\chi, \chi] = 1 + \sum_{i=2}^{k} r_i^2$. The only way to have $[\chi, \chi] = 2$ is if one r_i is 1 and the remaining r_i's are zero, which says that $\pi' \sim \pi_i$ is irreducible. $\square$

7.3 Example. Consider $G = A_4$ in its usual representation of degree 4 on $\Omega = \{1, 2, 3, 4\}$. This is a doubly transitive action, and the permutation character χ satisfies $\chi\big((1)\big) = 4$, $\chi\big((12)(34)\big) = 0$, $\chi\big((123)\big) = \chi\big((124)\big) = 1$. One verifies directly that $[\chi, \chi] = \frac{16}{12} + \frac{0}{4} + \frac{1}{3} + \frac{1}{3} = 2$. It follows that $\chi' = \chi - \chi_1$ is an irreducible character of degree 3, which of course is the character χ_4 of Example 6.3. But now we have a representation of G which affords χ_4. Starting with the permutation module $V = \mathbb{C}^4$ with standard basis $\{e_X : X \in \Omega\} = \{e_1, e_2, e_3, e_4\}$, we have $V = \langle v_1 \rangle \oplus U$ where $v_1 = (1, 1, 1, 1)$ and $U = \{(a_1, a_2, a_3, a_4) : \sum a_i = 0\}$. Here U is the required three-dimensional $\mathbb{C}G$-module, and if explicit 3×3 matrices $\pi_4(g)$ are required, these may be found just as in Example 1.1.

7.4 Example. We compute the character table of $G = A_5$. We know that G has exactly five conjugacy classes, with representatives (1), $(12)(34)$, (123), (12345) and (12354); the respective centralizer orders are 60, 4, 3, 5 and 5. Since G is a nonabelian simple group, G is perfect, so its only linear character is the principal character χ_1. Let χ be the permutation character of G in its usual (doubly transitive) representation on five points. By Theorem 7.2, $\chi_2 \stackrel{\mathrm{def}}{=} \chi - \chi_1$ is irreducible of degree 4. We have entered the values

of χ_2 in the table below. Now the degrees of the irreducible characters must satisfy $|G| = 60 = 1^2 + 4^2 + n_3^2 + n_4^2 + n_5^2$. It is easy to check that the remaining degrees can only be 3, 3 and 5.

Note that $G \cong PSL(2,5)$ acts 2-transitively on the 6 points of the projective line over $GF(5)$. In this representation, $(12)(34) \leftrightarrow \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ fixes two points $\left\langle \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) \right\rangle$ and $\left\langle \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right) \right\rangle$; similarly $(123) \leftrightarrow \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ fixes zero points, and every element of order 5, conjugate to $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ or to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, fixes exactly one point. By Theorem 7.2, subtracting off χ_1 from this permutation character, gives the irreducible character χ_3 of degree 5 with values as shown here:

| $|C_G(g)|$ | 60 | 4 | 3 | 5 | 5 |
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3				
χ_5	3				

It is now possible to complete the character table using the orthogonality relations (Theorems 5.5 and 5.7), and we leave this as an exercise. Instead, we produce the missing three-dimensional representations and determine the corresponding characters. To do this, we represent G as the group of rotational symmetries of a regular dodecahedron, with vertices labelled by the 20 ordered pairs (i,j) such that i, j are distinct members of $\{1, 2, 3, 4, 5\}$. This dodecahedron is as pictured:
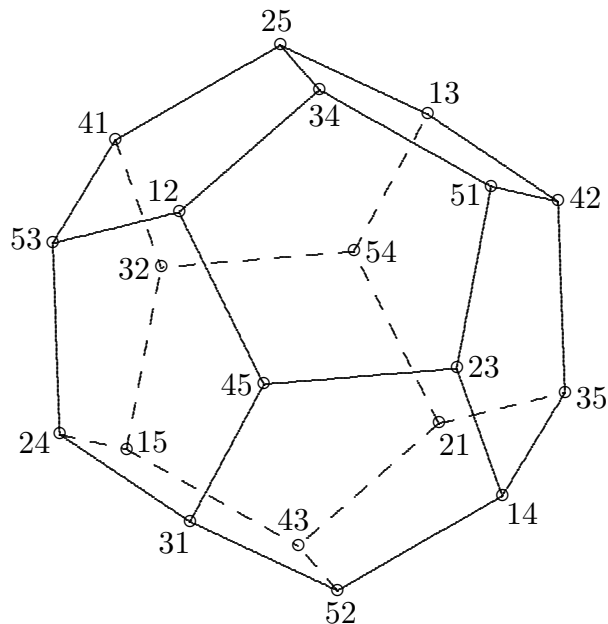
This representation of G must be irreducible over $\mathbb{C}$; for otherwise, it would decompose as a direct sum of three copies of the trivial representation, which would mean that G acts as the identity of order 3, a contradiction. So we have an irreducible representation

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$\pi_4 : G \to GL(3, \mathbb{R}) \subset GL(3, \mathbb{C})$, and we may suppose that χ_4 is its character, one of the irreducible characters of degree 3.

Now the effect of $(12)(34)$ is a $180°$ rotation about an axis which joins the midpoints of the edges 14—23 and 41—32. The remaining conjugacy class representatives also act as rotations about certain axes, through the angles indicated by the following pictures:

action of (123)   action of (12345)   action of (12354)
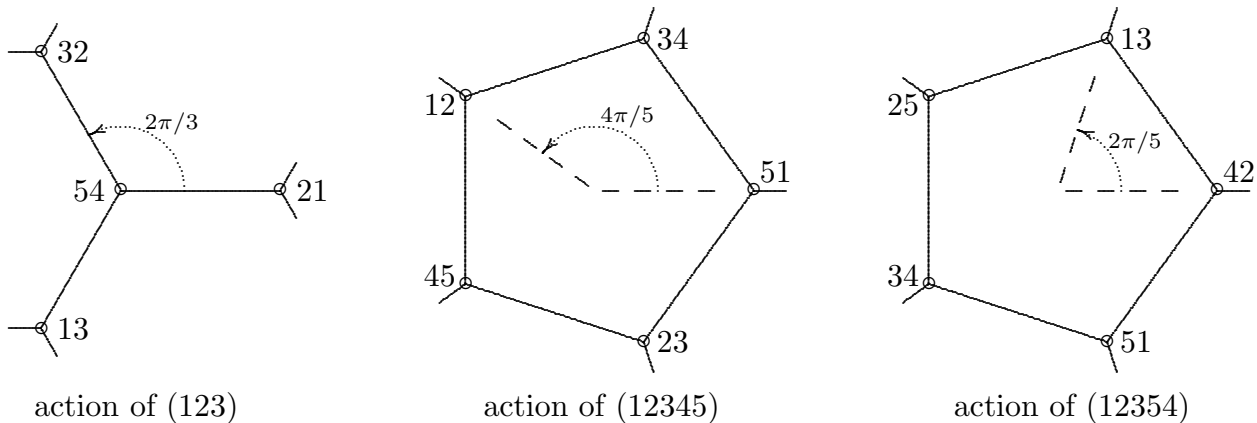
In general, any rotation about an axis through an angle θ may be represented (with respect to an appropriate basis) by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

whose trace is $1 + 2\cos\theta$. This gives the values of χ_4 given in the table below.

Finally, the values of the remaining irreducible character χ_5 are determined using the orthogonality relations. Alternatively, let $\sigma : G \to G$ be an outer automorphism, for example conjugation by (45). Then $g \mapsto \pi_4(g^\sigma)$ is also a homomorphism, and hence is an irreducible representation of degree 3, whose character is χ_5.

| $|C_G(g)|$ | 60 | 4 | 3 | 5 | 5 |
g	(1)	(12)(34)	(123)	(12345)	(12354)
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_5	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$

With the small groups we have looked at so far, we have been fortunate to discover most of the irreducible characters quickly, and then the remaining irreducible characters, if any, have been determined using the orthogonality relations. For larger groups we will not always be so lucky, and other methods must be used to complete the character table. In the next section, we will learn how to determine characters of a given group using the character tables of its proper subgroups.

8. Induced Characters

Given any character ψ of G and any subgroup $H \leq G$, the restriction of ψ to H, denoted ψ_H, is a character. This much is clear. (Be warned, however, that ψ might be irreducible without ψ_H being irreducible.)

Now consider the reverse problem: Given a character χ of $H \leq G$, does this 'lift' in any sense to a character of G? Warning: It is not true in general that every character of H is the *restriction* of a character of G. Therefore we cannot hope that 'lift' means to extend in the usual sense for functions. Nevertheless, there is a natural way, given χ, to produce an *induced character* of G, denoted χ^G. Moreover, the theory of induced characters is very important in understanding the subgroup structure of groups. For the present, however, we are content to motivate this topic by saying that it can be used to help determine the character table of a group from those of its proper subgroups.

Let χ be a character of H, where $H \leq G$. We first extend χ to all of G in the easiest possible way, by defining

$$\widehat{\chi}(g) = \begin{cases} \chi(g), & g \in G; \\ 0, & \text{otherwise.} \end{cases}$$

Now define the **'induced character'**

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \widehat{\chi}(x^{-1}gx), \qquad g \in G.$$

It is not yet clear that this is a character of G, which is why we used quotes ' ' above. In the next section, we will produce a representation which affords χ^G, to justify the terminology. For now, though, let us accept χ^G as the function defined above, and consider its properties. First, it is easy to see that $\chi^G : G \to \mathbb{C}$ is a class function. For if $g, y \in G$, then

$$\chi^G(g^y) = \frac{1}{|H|} \sum_{x \in G} \widehat{\chi}(g^{yx}) = \frac{1}{|H|} \sum_{x' \in G} \widehat{\chi}(g^{x'}) = \chi^G(g).$$

Since χ^G is a class function on G, it makes sense to write $[\chi^G, \psi]$ for any class function ψ of G. Now we must be careful: does $[\ ,\ ]$ mean inner product of class functions of G, or inner product of class functions of H? Although the context will usually prevent ambiguity, we will play it safe by using subscripts to indicate the correct space, thus:

$$[\psi, \psi']_G = \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\psi'(g)}$$

for any two class functions ψ, ψ' of G, and

$$[\theta, \theta']_H = \frac{1}{|H|} \sum_{h \in H} \theta(h)\overline{\theta'(h)}$$

for any two class functions θ, θ' of H. Also we denote the principal characters of H and G by 1_H and 1_G in order to distinguish them; thus $1_H(h) = 1$ and $1_G(g) = 1$ for all $h \in H$, $g \in G$.

Our first example of induced characters is something we have seen already: permutation characters!

8.1 Theorem. *Suppose G acts transitively on Ω. Then the permutation character is $(1_H)^G$ where $H = G_X$ is the stabilizer of any point $X \in \Omega$.*

Proof. The points of Ω are in one-to-one correspondence with the right cosets Hg, $g \in G$, and G acts on these right cosets by right multiplication, giving a transitive permutation action equivalent to the action on Ω. The premutation character is given by

$$
\begin{aligned}
\psi(g) &= \text{number of points of } \Omega \text{ fixed by } g \\
&= \text{number of right cosets } Hu \text{ such that } Hug = Hu \\
&= \frac{1}{|H|} \left| \{ u \in G \,:\, Hug = Hu \} \right| \\
&= \frac{1}{|H|} \left| \{ u \in G \,:\, ugu^{-1} \in H \} \right| \\
&= \frac{1}{|H|} \sum_{u \in G} 1_H(ugu^{-1}) \\
&= (1_H)^G(g).
\end{aligned}
$$
$\square$

8.2 Frobenius Reciprocity Theorem. *Let ψ be a character of G, and χ a character of $H \leq G$. Then*

$$
[\chi^G, \psi]_G = [\chi, \psi_H]_H \, .
$$

Proof.

$$
\begin{aligned}
[\chi_G, \psi]_G &= \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)} \\
&= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \widehat{\chi}(x^{-1}gx) \overline{\psi(g)} \\
&= \frac{1}{|G||H|} \sum_{u \in G} \sum_{x \in G} \widehat{\chi}(u) \overline{\psi(xux^{-1})} \, ,
\end{aligned}
$$

substituting $u = x^{-1}gx$. Now $\psi(xux^{-1}) = \psi(u)$ since ψ is a class function on G. Also $\widehat{\chi}(u)$ vanishes unless $u \in H$, so

$$
\begin{aligned}
[\chi^G, \psi]_G &= \frac{1}{|G||H|} \sum_{x \in G} \sum_{u \in H} \chi(u) \overline{\psi(u)} \\
&= \frac{1}{|H|} \sum_{u \in H} \chi(u) \overline{\psi_H(u)} = [\chi, \psi_H]_H \, .
\end{aligned}
$$
$\square$

As an example of how to apply the previous two results, let G act transitively on Ω, with point stabilizer $H = G_X$. Then the permutation character is $(1_H)^G$, and the multiplicity of the trivial representation 1_G in the permutation character is $[(1_H)^G, 1_G]_G = [1_H, 1_H] = 1$, a fact already known by Proposition 7.1.

For purposes of computing induced characters, calculations are simplified by choosing a **right transversal** T for H in G, which is a set of representatives of the distinct right cosets of H in G. Thus $|T| = [G : H]$ and $G = \bigcup_{t \in T} Ht$. Note that in general T is only a subset, not a subgroup, of G. Then for $g \in G$,

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \widehat{\chi}(xgx^{-1}) = \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \widehat{\chi}(htgt^{-1}h^{-1}).$$

But clearly $\widehat{\chi}(huh^{-1}) = \widehat{\chi}(u)$ for all $u \in G$, $h \in H$. This proves the following.

8.3 Lemma. $\chi^G(g) = \sum_{t \in T} \widehat{\chi}(tgt^{-1})$.

8.4 Example. For this example, *assume* that induced characters are characters. (We still have yet to prove this.) Let us determine the character table of $G = S_5$. First, $G' = A_5$, so G has exactly two linear characters, χ_1 and χ_2, corresponding to the two linear characters of $G/G' \cong C_2$. These, of course, are the principal character and the 'sign' character analogous to the situation of Example 1.1. Also, G has two doubly transitive permutation actions: one of degree 5, acting naturally as S_5; and the other as $PGL(2,5)$, acting on the 6 points of the projective line over $GF(5)$. Using Theorem 7.2, just as in Example 7.4, this gives irreducible characters χ_3 and χ_4 of degree 4 and 5 respectively. Details of these computations are left to the reader, and we have this much of the character table of G so far:

$|C_G(g)|$	120	12	8	6	4	5	6
g	(1)	(12)	$(12)(34)$	(123)	(1234)	(12345)	$(123)(45)$
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1
χ_3	4	2	0	1	0	-1	-1
χ_4	5	-1	1	-1	1	0	-1
χ_5							
χ_6							
χ_7							

The degrees of the remaining irreducible characters must satisfy $|G| = 120 = 1^1 + 1^2 + 4^2 + 5^2 + n_5^2 + n_6^2 + n_7^2$. There are two possible solutions for the missing degrees: 2,3,8 or 4,5,6. In this instance the orthogonality relations are not adequate to determine the rest of the table. Instead, we induce characters from the subgroup $H = A_5$. We have a right transversal $T = \{(1), (12)\}$, and so for each character χ of H, we obtain an induced character $\chi^G(g) = \widehat{\chi}(g) + \widehat{\chi}(g^{(12)})$ (from Lemma 8.3, instead of the definition, which would give 120 terms). Let χ be one of the irreducible characters of H of degree 3 (say, χ_4 of Example 7.4). Then we obtain an induced character χ^G with values

g	(1)	(12)	$(12)(34)$	(123)	(1234)	(12345)	$(123)(45)$
$\chi^G(g)$	6	0	-2	0	0	1	0

Of course there is no guarantee that an induced character is irreducible. However, if $\chi^G = \sum_i r_i \chi_i$ then $[\chi^G, \chi^G] = \sum_i r_i^2$. In this case we easily compute $[\chi^G, \chi^G] = 1$. Therefore χ^G is irreducible, and we may add it to our table. Next let η be the irreducible character of H of degree 4 (called χ_2 in Example 7.4). In the same way, we compute the following values for the induced character η^G:

g	(1)	(12)	(12)(34)	(123)	(1234)	(12345)	(123)(45)
$\eta^G(g)$	8	0	0	2	0	-2	0

Now we determine $[\eta^G, \eta^G] = 2$. The only way to have $\sum_i r_i^2 = 2$ is if two r_i's are 1 and the rest zero; therefore η^G is the sum of two distinct irreducible characters of G. It is easy to check that $[\eta^G, \chi_i] = 0$ for $i = 1, 2$. But $[\eta^G, \chi_3] = 1$, so η^G is the sum of χ_3 and another irreducible character of degree 4, say χ_6. This gives the values of $\chi_6 = \eta^G - \chi_3$, and the remaining irreducible character χ_7 is determined by the orthogonality relations. Finally, the character table of G is as follows:

$|C_G(g)|$	120	12	8	6	4	5	6
g	(1)	(12)	(12)(34)	(123)	(1234)	(12345)	(123)(45)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	-1	1	-1
χ_3	4	2	0	1	0	-1	-1
χ_4	5	-1	1	-1	1	0	-1
χ_5	6	0	-2	0	0	1	0
χ_6	4	-2	0	1	0	-1	1
χ_7	5	1	1	-1	-1	0	1

Finally, let us give a further illustration of Frobenius reciprocity using the above notation: we may directly compute

$$[\eta^G, \chi_i]_G = \begin{cases} 1, & i = 3 \text{ or } 6 \\ 0, & \text{otherwise} \end{cases}$$

and this agrees with $[\eta, (\chi_i)_H]_H$, which is also easy to compute using the information of Example 7.4.

9. Induced Representations

Now we show how representations of a subgroup $H \leq G$ may be 'induced' to representations of G. The character associated to such an induced representation will turn out to be the induced character defined in the previous section, and this provides the justification for calling the χ^G of Section 8 a character.

Let $H \leq G$, and choose a right transversal T of H in G, so that $|T| = [G:H]$ and $G = \bigcup_{t \in T} Ht$, as we did prior to Lemma 8.3. Let $\pi : H \to GL(V)$ be a representation of H. We assume that $n = \dim V$ is finite, but it is not necessary to place any restrictions on the field F. We consider V as not only a vector space over F, but also a module for the group algebra FG.

Now the usual tensor product of V with FG (over F) gives a vector space of dimension $(\dim V)(\dim FG) = n|G|$, denoted $V \otimes FG$ or $V \otimes_F FG$. If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\{v_i \otimes g \ : \ 1 \leq i \leq n, \ g \in G\}$ is a basis for $V \otimes_F FG$. We make $V \otimes_F FG$ into an FG-module by defining $(v \otimes g)a = v \otimes (ga)$ for $v \in V$, $g \in G$, $a \in FG$.

We are more interested in the tensor product of V and FG over FH, denoted $V \otimes_{FH} FG$. This is obtained as a quotient of $V \otimes_F FG$ in which we identify $v \otimes hg = vh \otimes g$ for all $v \in V$, $g \in G$, $h \in H$. (Note: vh means h acts on the vector v via π.) Since every $g \in G$ can be written uniquely as $g = ht$ for some $h \in H$ and $t \in T$, we may write $v \otimes g = v \otimes ht = vh \otimes t$. Thus $\{v_i \otimes t \ : \ 1 \leq i \leq n, \ t \in T\}$ is a basis for $V \otimes_{FH} FG$, and we have $\dim(V \otimes_{FH} FG) = n[G:H]$. We define the **induced module** as $V^G = V \otimes_{FH} FG$, with the action of FG defined above. The **induced representation** is the corresponding representation $\pi^G : G \to GL(V^G)$.

9.1 Example. Consider the representation π_3 of $H = S_3$ defined in Example 1.1. Consider H as a subgroup of $G = S_4$, with right transversal $T = \{(1), (12)(34), (13)(24), (14)(23)\}$. (Note: In general we cannot hope that T is a subgroup of G.) We will obtain the induced representation π_3^G, of degree $2[G:H] = 8$. Let $\{e_1, e_2\}$ be the standard basis of $V = \mathbb{C}^2$, the module corresponding to π_3. Then

$$\big(e_1 \otimes (1)\big)\,(12) = e_1 \otimes (1)(12) = e_1 \otimes (12) = e_1(12) \otimes (1) = e_2 \otimes (1),$$
$$\big(e_2 \otimes (1)\big)\,(12) = e_2 \otimes (1)(12) = e_2 \otimes (12) = e_2(12) \otimes (1) = e_1 \otimes (1),$$
$$\big(e_1 \otimes (12)(34)\big)(12) = e_1 \otimes (12)(34)(12) = e_1 \otimes (12)(12)(34) = e_1(12) \otimes (12)(34)$$
$$= e_2 \otimes (12)(34),$$
$$\big(e_2 \otimes (12)(34)\big)(12) = e_2 \otimes (12)(34)(12) = e_2 \otimes (12)(12)(34) = e_2(12) \otimes (12)(34)$$
$$= e_2 \otimes (12)(34),$$

$$
\begin{aligned}
\big(e_1 \otimes (13)(24)\big)(12) &= e_1 \otimes (13)(24)(12) = e_1 \otimes (12)(14)(23) = e_1(12) \otimes (14)(23) \\
&= e_2 \otimes (14)(23), \\
\big(e_2 \otimes (13)(24)\big)(12) &= e_2 \otimes (13)(24)(12) = e_2 \otimes (12)(14)(23) = e_2(12) \otimes (14)(23) \\
&= e_1 \otimes (14)(23), \\
\big(e_1 \otimes (14)(23)\big)(12) &= e_1 \otimes (14)(23)(12) = e_1 \otimes (12)(13)(24) = e_1(12) \otimes (13)(24) \\
&= e_2 \otimes (13)(24), \\
\big(e_2 \otimes (14)(23)\big)(12) &= e_2 \otimes (14)(23)(12) = e_2 \otimes (12)(13)(24) = e_2(12) \otimes (13)(24) \\
&= e_1 \otimes (13)(24).
\end{aligned}
$$

This shows that with respect to the ordered basis $\{e_1\otimes(1),\ e_2\otimes(1),\ \ldots,\ e_2\otimes(14)(23)\}$, we have the matrix

$$
\pi_3^G\big((12)\big) = \begin{pmatrix}
\boxed{\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}} & & & \\
& \boxed{\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}} & & \\
& & & \boxed{\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}} \\
& & \boxed{\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}} &
\end{pmatrix}.
$$

Note the block form of this matrix. The blocks are in a 4×4 permutation matrix pattern, corresponding to the way (12) permutes the four right cosets of H. Similar computations give the remaining matrices $\pi_3^G(g)$. We present here just a few more examples:

$$
\pi_3^G\big((1)\big) = \begin{pmatrix}
\boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & & & \\
& \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & & \\
& & \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & \\
& & & \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}}
\end{pmatrix}, \quad
\pi_3^G\big((123)\big) = \begin{pmatrix}
\boxed{\begin{smallmatrix}\omega & 0\\ 0 & \overline{\omega}\end{smallmatrix}} & & & \\
& & & \boxed{\begin{smallmatrix}\omega & 0\\ 0 & \overline{\omega}\end{smallmatrix}} \\
& \boxed{\begin{smallmatrix}\omega & 0\\ 0 & \overline{\omega}\end{smallmatrix}} & & \\
& & \boxed{\begin{smallmatrix}\omega & 0\\ 0 & \overline{\omega}\end{smallmatrix}} &
\end{pmatrix},
$$

$$
\pi_3^G\big((12)(34)\big) = \begin{pmatrix}
& & \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & \\
\boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & & & \\
& & & \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} \\
& \boxed{\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}} & &
\end{pmatrix}, \quad
\pi_3^G\big((1234)\big) = \begin{pmatrix}
& & & \boxed{\begin{smallmatrix}0 & \overline{\omega}\\ \omega & 0\end{smallmatrix}} \\
\boxed{\begin{smallmatrix}0 & \overline{\omega}\\ \omega & 0\end{smallmatrix}} & & & \\
& \boxed{\begin{smallmatrix}0 & \overline{\omega}\\ \omega & 0\end{smallmatrix}} & & \\
& & \boxed{\begin{smallmatrix}0 & \overline{\omega}\\ \omega & 0\end{smallmatrix}} &
\end{pmatrix}.
$$

From these we may obtain the associated character χ^G, which has values given by

g	(1)	(12)	$(12)(34)$	(123)	(1234)
$\chi^G(g)$	8	0	0	-1	0

As an exercise, you can check that this agrees with the induced character χ^G defined in Section 8, where χ is the character associated with π_3. The next result shows that this works in general.

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9.2 Theorem. *Let $\pi : H \to GL(V)$, where $H \leq G$, and let χ be the character associated with π. Then the character associated with π^G is the induced character χ^G defined in Section 8.*

Proof. Recall that $\{v_i \otimes t : 1 \leq i \leq n,\ t \in T\}$ is a basis for V^G, in our earlier notation. For each $g \in G$, we have $(v \otimes t)g = v \otimes tg = v \otimes ht' = vh \otimes t'$, where we have written $tg = ht'$ for some $h \in H$ and $t' \in T$ uniquely determined by g and t. Thus g (or more precisely, $\pi^G(g)$) maps $V \otimes t$ to $V \otimes t'$, where $V \otimes t = \{v \otimes t : v \in V\}$. Now V^G is a direct sum of the subspaces $V \otimes t$, each of dimension n, so the matrix of $\pi^G(g)$ consists of $n \times n$ blocks in a permutation pattern, as in Example 9.1. The only blocks which contribute to $\operatorname{tr} \pi^G(g)$ are those on the diagonal, i.e. those for which $t' = t$. Equivalently, those for which $h = tgt^{-1} \in H$. For such g and t, we have $(v \otimes t)g = v(tgt^{-1}) \otimes t$, and the trace of the corresponding $n \times n$ block is $\operatorname{tr} \pi(tgt^{-1}) = \chi(tgt^{-1})$. The total trace of $\pi^G(g)$ is the total contribution from such diagonal blocks, which is $\sum_{t\,:\,tgt^{-1} \in H} \chi(tgt^{-1}) = \sum_{t \in T} \widehat{\chi}(tgt^{-1}) = \chi^G(g)$. $\qquad\square$

10. Frobenius Groups

In this section we present an application of character theory to Frobenius groups. However, we begin with just a few further facts about characters and representations.

The kernel of a representation $\pi : G \to GL(n, \mathbb{C})$ is $\{g \in G : \pi(g) = I\}$. Interestingly, it is possible to tell whether a given $g \in G$ lies in $\ker \pi$ simply by examining its trace, $\chi(g) = \operatorname{tr} \pi(g)$. For if $\pi(g) = I$ (the $n \times n$ identity) then $\chi(g) = n$, the degree of the representation. Conversely, suppose that $\chi(g) = n$. Recall from the proof of Lemma 5.1 that $\pi(g)$ is similar to $\operatorname{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where each ϵ_i is a root of unity, and $\chi(g) = \sum_i \epsilon_i$. Evidently the only way to have $\sum_{i=1}^{n} \epsilon_i = n$ is if every $\epsilon_i = 1$, in which case $\pi(g)$ is similar to I, so $\pi(g) = I$. This proves the following.

10.1 Proposition. *Let $\pi : G \to GL(n, \mathbb{C})$ be a represetnation of a finite group G, and let χ be its associated character. Then $\pi(g) = I$ if and only if $\chi(g) = n$.*

We define $\ker \chi = \{g \in G : \chi(g) = n\}$, the **kernel** of the character χ. Since χ is not a homomorphism for $n > 1$, this is not the usual usage of the word 'kernel'; however since $\ker \chi = \ker \pi$, it is the kernel of *some* homomorphism (namely, π) and as such, it is a normal subgroup of G. We can get several normal subgroups of G by taking intersections of various $\ker \chi$ for certain choices of irreducible characters χ. It may not be obvious that *every* normal subgroup of G can be obtained in this way, but we will show this.

First, we show that $\bigcap_{\chi \in \operatorname{Irr}_G} \ker \chi = 1$. To see this, suppose that $1 \neq g \in G$. Then g and 1 lie in distinct conjugacy classes of G, so by Corollary 5.7, the columns of the character table of G corresponding to these two conjugacy classes are perpendicular, hence linearly independent, and hence distinct. So $\chi(g) \neq \chi(1) = n$ for some $\chi \in \operatorname{Irr}_G$.

Next, suppose that K is any normal subgroup of G. By a homework assignment, every irreducible character of G/K gives an irreducible character of G. (For each $\psi \in \operatorname{Irr}_{G/K}$, we define $\psi'(g) = \psi(gK)$. Then ψ' is well-defined and $\psi' \in \operatorname{Irr}_G$.) By the previous paragraph, we have $\bigcap_{\psi \in \operatorname{Irr}_{G/K}} \ker \psi = K = \bigcap_{\psi \in \operatorname{Irr}_{G/K}} \ker \psi'$. This proves the following.

10.2 Proposition. *Every normal subgroup of G is of the form $\bigcap \ker \chi$ as χ ranges over some subset of Irr_G.*

We are now ready for Frobenius groups. Let G be a permutation group acting on a finite set Ω of cardinality $n > 1$, i.e. $G \leq \mathrm{Sym}\,\Omega \cong S_n$. We call G a **Frobenius group** if
   (i) G acts transitively on Ω,
   (ii) the stabilizer of a point of Ω is nontrivial, and
   (iii) every nonidentity element of G fixes at most one point.
Since every transitive permutation action is equivalent to the action on the right cosets of the point stabilizer, we may give an alternative definition of Frobenius groups which is completely internal to G, without regard to Ω. Thus G is a Frobenius group if it has a proper nontrivial subgroup H such that $H \cap H^g = 1$ for all $g \in G \smallsetminus H$. To see that the second definition follows from the first, let G be a Frobenius group acting on Ω, and let $H = G_X$, the stabilizer of a point $X \in \Omega$ (chosen arbitrarily). Then H is a proper subgroup of G since $[G : H] = |\Omega| = n > 1$, and $H \neq 1$ by (ii). If $g \in G \smallsetminus H$, then $X^g \neq X$ and the stabilizer of $X^g \in \Omega$ is $H^g = g^{-1}Hg$. Condition (iii) says that only the identity fixes both X and X^g, so $H \cap H^g = 1$. The converse is proven similarly. It may seem possible that a given group G might be a Frobenius group relative to more than one choice of subgroup H. However, this does not happen: if G is a Frobenius group, then there exists (up to conjugacy) a unique subgroup H satisfying the above condition, and so G has a unique (up to equivalence) permutation representation as a Frobenius group. We will not prove this fact here.

The following is the main result of this section.

10.3 Theorem (Frobenius). *Let G be a Frobenius group acting on Ω. Then $G = K \rtimes H$ where H is a point stabilizer, and*

$$K = \left(G \smallsetminus \bigcup_{g \in G} H^g \right) \cup \{1\} = \{ g \in G : g \text{ fixes no points of } \Omega \} \cup \{1\}.$$

We call K (as above) the **Frobenius kernel** and H the **Frobenius complement**. Note that it is easy to define K (and in fact we have given two equivalent conditions of K). The hard part is showing that this K gives a *subgroup* of G, and amazingly, the only known proof of this uses character theory! Indeed, it is possible to prove much more using character theory. For example, K must be nilpotent, and there are strong restrictions on H as well. We will not prove these further results here.

Before proceeding with a proof of Theorem 10.3, we require the following.

10.4 Lemma. *Let θ be a class function of H such that $\theta(1) = 0$. Then $(\theta^G)_H = \theta$.*

Proof. By definition, for each $g \in G$ we have $(\theta^G)_H(g) = \frac{1}{|H|} \sum_{x \in G} \widehat{\theta}(xgx^{-1})$. In particular,

$$(\theta^G)_H(1) = \frac{1}{|H|} \sum_{x \in G} \theta(1) = 0.$$

If $1 \neq h \in H$, then $(\theta^G)_H(h) = \frac{1}{|H|}\sum_{x \in G} \widehat{\theta}(xhx^{-1})$ only has nonzero values for $x \in H$. Then

$$(\theta^G)_H(h) = \frac{1}{|H|}\sum_{x \in H} \theta(xhx^{-1}) = \frac{1}{|H|}\cdot|H|\theta(h) = \theta(h)\,.$$

$\square$

Proof of Theorem 10.3. Condition (iii) above shows that $N_G(H) = H$. Now the number of conjugates of H in G is $[G : N_G(H)] = [G : H] = n$, and these conjugates intersect in only the identity, so

$$|K| = |G| - [G:H]\big(|H|-1\big) = |G| - |G| + [G:H] = [G:H].$$

Choose any nonprincipal character $\psi \in \mathrm{Irr}_H$. Define a class function of H by $\theta = \psi - \psi(1)1_H$, so that $\theta(1) = 0$. Then Lemma 10.4 gives $(\theta^G)_H = \theta$. Now define $\psi^* = \theta^G + \psi(1)1_G$. Then by the Frobenius Reciprocity Theorem 8.2, we have

$$\begin{aligned}
[\psi^*, \psi^*]_G &= [\theta^G, \theta^G]_G + 2\psi(1)[\theta^G, 1_G]_G + \psi(1)^2[1_G, 1_G]_G \\
&= [\theta, (\theta^G)_H]_H + 2\psi(1)[\theta, 1_H]_H + \psi(1)^2 \\
&= [\theta, \theta]_H + 2\psi(1)\big(-\psi(1)\big) + \psi(1)^2 \\
&= 1 + \psi(1)^2 - 2\psi(1)^2 + \psi(1)^2 \\
&= 1\,.
\end{aligned}$$

However, $\psi^* = \theta^G + \psi(1)1_G$ is a $\mathbb{Z}$-linear combination of characters, so $\pm\psi^* \in \mathrm{Irr}_G$. But if $h \in H$, then $\psi^*(h) = \theta^G(h) + \psi(1) = \theta(h) + \psi(1) = \psi(h)$. This means that $(\psi^*)_H = \psi$, and in particular, $\psi^*(1) = \psi(1) > 0$, and so $\psi^* \in \mathrm{Irr}_G$.

Define $M = \bigcap_{\psi \in \mathrm{Irr}_H} \ker \psi^*$. Then $M \trianglelefteq G$. We will show that $M = K$. If $h \in M \cap H$, then for every nonprincipal character $\psi \in \mathrm{Irr}_H$, we have $\psi(h) = \psi^*(h) = \psi^*(1) = \psi(1)$. By the arguments preceding Proposition 10.2, this implies that $h = 1$. Since $M \trianglelefteq G$, for every $g \in G$ we obtain $M \cap H^g = (M \cap H)^g = 1^g = 1$, and so $M \subseteq K$. Conversely, if $1 \neq g \in K$ (i.e. g lies in no conjugate of H) then $\theta^G(g) = \frac{1}{|H|}\sum_{x \in G} \widehat{\theta}(xgx^{-1}) = 0$, and so $\psi^*(g) = \theta^G(g) + \psi(1) = \psi(1) = \psi^*(1)$, i.e. $g \in \ker \psi^*$ for all $\psi \in \mathrm{Irr}_H$. This means that $g \in M$, i.e. $K \subseteq M$, and so as claimed, $K = M$, a normal subgroup of G.

Finally, $|KH| = |K||H|/|K \cap H| = |K||H| = |G|$ and so $G = KH = K \rtimes H$. $\square$

11. The Center of the Group Algebra

Let G be a finite group, with conjugacy classes $K_1 = \{1\}$, K_2, ..., K_k; irreducible representations π_1, π_2, ..., π_k, and irreducible characters $\mathrm{Irr}_G = \{\chi_1 = 1_G, \chi_2, \ldots, \chi_k\}$. As seen before, the center of the group algebra, $Z(\mathbb{C}G)$, is k-dimensional. Namely, $Z(\mathbb{C}G)$ is the set of all $\sum_{g \in G} a_g g$ such that the coefficients $a_g \in \mathbb{C}$ are constant on conjugacy classes. So a basis for $Z(\mathbb{C}G)$ is $\{\gamma_1 = 1, \gamma_2, \ldots, \gamma_k\}$ where $\gamma_i = \sum_{g \in K_i} g$. As we might anticipate from Theorem 4.4, character theory has much to tell about $\mathbb{C}G$ and conjugacy classes. Especially, we will use character theory to answer the question: can we write an element $g \in K_\ell$ as a product xy with $x \in K_i$ and $y \in K_j$? If so, in how many ways is this possible? Note that the answers to these questions cannot depend on which representative $g \in K_\ell$ we choose, since conjugation acts transitively on each conjugacy class. The answers can only depend on i, j and ℓ.

Let $a_{ij\ell}$ be the number of ways each $g \in K_\ell$ can be written as xy with $x \in K_i$ and $y \in K_j$. So the constants $a_{ij\ell}$ are non-negative integers, which we must evaluate. Also it is clear that

$$(11.1) \qquad \gamma_i \gamma_j = \sum_{\ell=1}^{k} a_{ij\ell} \gamma_l \,.$$

So the structure of the algebra $Z(\mathbb{C}G)$ is completely determined by the constants $a_{ij\ell}$, since $Z(\mathbb{C}G)$ is first of all a k-dimensional vector space over $\mathbb{C}$; and secondly, $Z(\mathbb{C}G)$ is a ring with products defined by (11.1). So the constants $a_{ij\ell}$ are called **structure constants** of the algebra $Z(\mathbb{C}G)$.

Consider any matrix representation $\pi : G \to GL(n, \mathbb{C})$. Then there is a unique extension of π to an algebra homomorphism, also denoted π, namely

$$\pi : \mathbb{C}G \to M(n, \mathbb{C}), \qquad \pi\Big(\sum_{g \in G} a_g g\Big) = \sum_{g \in G} a_g \pi(g) \,.$$

It is easy to check that this extension preserves multiplication:

$$\pi\Big(\Big(\sum_{g \in G} a_g g\Big)\Big(\sum_{h \in G} b_h h\Big)\Big) = \pi\Big(\sum_{g \in G}\sum_{h \in G} a_g b_h gh\Big) = \sum_{g \in G}\sum_{h \in G} a_g b_h \pi(gh)$$

$$= \sum_{g \in G}\sum_{h \in G} a_g b_h \pi(g)\pi(h) = \Big(\sum_{g \in G} a_g \pi(g)\Big)\Big(\sum_{h \in H} b_h \pi(h)\Big)$$

$$= \pi\Big(\sum_{g \in G} a_g g\Big)\pi\Big(\sum_{h \in H} b_h h\Big);$$

and similarly, π is $\mathbb{C}$-linear, so in fact $\pi : \mathbb{C}G \to M(n, \mathbb{C})$ is an algebra homomorphism.

Now suppose that $\alpha \in \mathbb{C}G$ and $z \in Z(\mathbb{C}G)$. Then $\pi(z)\pi(\alpha) = \pi(z\alpha) = \pi(\alpha z) = \pi(\alpha)\pi(z)$. This says that $\pi(z)$ commutes with all matrices $\pi(a)$ for $a \in \mathbb{C}G$, and in

particular $\pi(z)$ commutes with all matrices $\pi(g)$ for $g \in G$. By Schur's Lemma 3.2, $\pi(z)$ is a scalar multiple of the $n \times n$ identity matrix I, so

$$\pi(z) = \omega_\chi(z)I \qquad \text{for all } z \in Z(\mathbb{C}G).$$

Here we have called the scalar $\omega_\chi(z) \in \mathbb{C}$. We could have called it $\omega_\pi(z)$, since it depends on the choices of both $z \in Z(\mathbb{C}G)$ and the representation π; however, since π is uniquely determined by its associated character χ (see Corollary 5.6), it is customary to instead write $\omega_\chi(g)$. Since $\pi : \mathbb{C}G \to M(n,\mathbb{C})$ is an algebra homomorphism, it is easy to see that $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ is an algebra homomorphism. The values of ω_χ are determined by the values on the basis $\{\gamma_i\}$ of $Z(\mathbb{C}G)$. These values determined by the following.

11.2 Proposition. *For any character χ of G, and any class sum γ_i, the value of $\omega_\chi(\gamma_i)$ is an algebraic integer given by*

$$\omega_\chi(\gamma_i) = \frac{|K_i|\chi(g_i)}{\chi(1)}$$

where $g_i \in K_i$.

Proof. Take traces on both sides of the equation $\pi(\gamma_i) = \omega_\chi(\gamma_i)I$. The left side gives $\operatorname{tr}\pi(\gamma_i) = |K_i|\operatorname{tr}\pi(g_i) = |K_i|\chi(g_i)$. The right side gives $\omega_\chi\operatorname{tr}I = \omega_\chi\chi(1)$ since I is a square identity matrix of size $\deg\pi = \chi(1)$. This gives the explicit formula for $\omega_\chi(\gamma_i)$ given above.

Applying the algebra homomorphism $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$ to (11.1), we obtain

$$(11.3) \qquad \omega_\chi(\gamma_i)\omega_\chi(\gamma_j) = \sum_{\ell=1}^{k} a_{ij\ell}\omega_\chi(\gamma_l).$$

Consider the $k \times k$ matrix $A = \big(a_{ij\ell} : 1 \le j, \ell \le k\big)$, where i is fixed, $1 \le i \le k$. According to the above,

$$Av = \omega_\chi(\gamma_i)v, \qquad \text{where } v = \begin{pmatrix} \omega_\chi(\gamma_1) \\ \omega_\chi(\gamma_2) \\ \vdots \\ \omega_\chi(\gamma_k) \end{pmatrix}.$$

Let $f(X) = det(XI - A)$, the characteristic polynomial of A. Then $f(A)$ is the zero matrix, so $f(\omega_\chi(\gamma_i))v = f(A)v = 0$. Of course not all entries of v are zero; for example, the first entry is $\omega_\chi(1) = |\{1\}|\chi(1)/\chi(1) = 1$. Therefore $f(\omega_\chi(\gamma_i)) = 0$. So $\omega_\chi(\gamma_i)$ is a root of a monic polynomial $f(X)$ having integer coefficients, since the entries of A are integers. Thus $\omega_\chi(\gamma_i)$ is an algebraic integer. $\qquad\square$

11.4 Corollary. *The structure constants of $Z(\mathbb{C}G)$ are given by*

$$a_{ij\ell} = \frac{|K_i||K_j|}{|G|} \sum_{\chi \in \operatorname{Irr}_G} \frac{\chi(g_i)\chi(g_j)\overline{\chi(g_\ell)}}{\chi(1)}.$$

Proof. Replace $\omega_\chi(\gamma_i) = |K_i|\chi(g_i)/\chi(1)$, and similarly for j, ℓ in (11.3) to obtain

$$|K_i||K_j|\frac{\chi(g_i)\chi(g_j)}{\chi(1)^2} = \sum_{\ell=1}^{k} a_{ij\ell}|K_\ell|\frac{\chi(g_\ell)}{\chi(1)}\,.$$

Multiply both sides by $\chi(1)\overline{\chi(g_s)}$ and sum over $\chi \in \mathrm{Irr}_G$ to obtain

$$|K_i||K_j|\sum_{\chi \in \mathrm{Irr}_G}\frac{\chi(g_i)\chi(g_j)\overline{\chi(g_s)}}{\chi(1)} = \sum_{\ell=1}^{k}a_{ij\ell}|K_\ell|\sum_{\chi \in \mathrm{Irr}_G}\chi(g_\ell)\overline{\chi(g_s)}$$

$$= \sum_{\ell=1}^{k}a_{ij\ell}|K_\ell|\frac{|G|}{|K_\ell|}\delta_{\ell s} = |G|a_{ijs}\,,$$

using the column orthogonality relations, Corollary 5.7. $\qquad\square$

11.5 Corollary. *The degrees of the irreducible representations of G divide $|G|$.*

Proof. Let χ be an irreducible character of G. From $[\chi, \chi] = 1$ we obtain

$$|G| = \sum_{i=1}^{k}|K_i|\chi(g_i)\overline{\chi(g_i)} = \chi(1)\sum_{i=1}^{k}\omega_\chi(\gamma_i)\overline{\chi(g_i)}\,.$$

Thus

$$\frac{|G|}{\chi(1)} = \sum_{i=1}^{k}\omega_\chi(\gamma_i)\overline{\chi(g_i)}\,.$$

Since the set of algebraic integers form a ring, $|G|/\chi(1)$ is an algebraic integer. However, $|G|/\chi(1)$ is rational; therefore $|G|/\chi(1) \in \mathbb{Z}$. That is, $\chi(1)$, which is the degree of the corresponding irreducible representation, divides $|G|$. $\qquad\square$

12. Burnside's Theorem

We come to another famous application of characcther theory: a theorem of Burnside which states that any group of order $p^a q^b$ (for primes p, q) is solvable. We need only a little more terminology and preparation to prove this.

Let $\pi : G \to GL(n, \mathbb{C})$ be an irreducible representation of a finite group G. When does $\pi(g)$ commute with $\pi(h)$ for all $h \in G$? By Schur's Lemma 3.2, this can only happen if $\pi(g)$ is a scalar multiple of I. It is possible to detect whether this happens for a given $g \in G$ directly from the values of the associated character χ. Recall that $\pi(g)$ is similar to $\mathrm{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where each ϵ_i is a root of unity. The only way this can be a scalar multiple of I is if $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n$, and this is equivalent to $\left|\sum_i \epsilon_i\right| = n$. This immediately gives

12.1 Lemma. *If π is an irreducible representation of G with associated character χ, then $\pi(g)$ commutes with $\pi(G)$ if and only if $\big|\chi(g)\big| = \chi(1)$.*

Accordingly, we define the **center** of χ to be $Z(\chi) = \big\{g \in G : |\chi(g)| = \chi(1)\big\}$.

12.2 Lemma. *If $\chi \in \mathrm{Irr}_G$, then $g \in Z(\chi)$ if and only if $g \ker \chi \in Z(G/\ker \chi)$. Thus $Z(\chi)/\ker \chi = Z\big(G/\ker \chi\big)$ and $Z(\chi) \trianglelefteq G$.*

Proof. The homomorphism $\pi : G \to \pi(G) < GL(n, \mathbb{C})$ has kernel $\ker \pi = \ker \chi$, so $\pi(G) \cong G/\ker \chi$. Using this isomorphism and Lemma 12.1, we have $g \in Z(\chi)$ if and only if $\pi(g) \in Z\big(\pi(G)\big)$, if and only if $g \ker \chi \in Z\big(G/\ker \chi\big)$. $\qquad\square$

The heart of Burnside's $p^a q^b$ theorem is the following. For this we need just a little Galois Theory. If $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are roots of unity, they are all powers of some root of unity, say ϵ where $\epsilon^m = 1$. (We may take m to be the least common multiple of $m_1, m_2, \ldots, m_n$ where $\epsilon_i^{m_i} = 1$.) The field $\mathbb{Q}(\epsilon_1, \ldots, \epsilon_n) = \mathbb{Q}(\epsilon)$ is a Galois extension of $\mathbb{Q}$, and it has exactly $\phi(m) = [\mathbb{Q}(\epsilon) : \mathbb{Q}]$ automorphisms. We denote these automorphisms by σ_s for those integers s such that $1 \leq s \leq m$ and $(s, m) = 1$. (The number of such integers denoted $\phi(m)$, and ϕ is known as Euler's function.) Each σ_s fixes every element of $\mathbb{Q}$, and maps $\epsilon \mapsto \epsilon^s$. Since every element of $\mathbb{Q}(\epsilon)$ is of the form $f(\epsilon)$ for some polynomial $f(X) \in \mathbb{Q}[X]$, the action of σ_s on $\mathbb{Q}(\epsilon)$ is determined by $f(\epsilon) \mapsto f(\epsilon^s)$. These automorphisms form a group $\mathrm{Gal}\big(\mathbb{Q}(\epsilon)/\mathbb{Q}\big) = \{\sigma_s : 1 \leq s \leq m, \ (s, m) = 1\}$, called the *Galois group* of the extension field $Q(\epsilon)$. By the Fundamental Theorem of Galois Theory, the only elements of $\mathbb{Q}(\epsilon)$ fixed by all σ_s are the rationals $x \in \mathbb{Q}$.

12.3 Theorem (Burnside). *Let $\chi \in \mathrm{Irr}_G$, and let $g \in K_i$ where K_i is a conjugacy class of G. Suppose that $\big(\chi(g), |K_i|\big) = 1$. Then either $g \in Z(\chi)$ or $\chi(g) = 0$.*

Proof. We may choose $u, v \in \mathbb{Z}$ such that $u\chi(1) + v|K_i| = 1$. Then

$$\frac{\chi(g)}{\chi(1)} = u\chi(g) + v\frac{|K_i|\chi(g)}{\chi(1)} \ .$$

Since $\chi(g)$ and $|K_i|\chi(g)/\chi(1)$ are algebraic integers (see Proposition 11.2), it follows that $\alpha = \chi(g)/\chi(1)$ is an algebraic integer. Suppose that $g \notin Z(\chi)$. Then $|\chi(g)| < \chi(1)$, so $|\alpha| < 1$. We must show that $\chi(g) = 0$, or equivalently that $\alpha = 0$.

As usual, we have $\chi(g) = \sum_{i=1}^n \epsilon_i$ where the ϵ_i's are roots of unity. Since $g \notin Z(\chi)$, the ϵ_i are not all the same. Let σ be an automorphism of $\mathbb{Q}(\epsilon_1, \ldots, \epsilon_n)$. Then $\epsilon_1^\sigma, \ldots, \epsilon_n^\sigma$ are roots of unity, not all the same, so $\big|\chi(g)^\sigma\big| = \big|\sum_i \epsilon_i^\sigma\big| < n = \chi(1)$. This proves that $|\alpha^\sigma| < 1$ for every $\sigma \in \mathrm{Gal}\big(\mathbb{Q}(\epsilon)/\mathbb{Q}\big)$. Define $\beta = \prod_\sigma \alpha^\sigma \in \mathbb{Q}(\epsilon)$, where the product ranges over all $\sigma \in \mathrm{Gal}\big(\mathbb{Q}(\epsilon)/\mathbb{Q}\big)$. For any $\tau \in \mathrm{Gal}\big(\mathbb{Q}(\epsilon)/\mathbb{Q}\big)$, we have $\beta^\tau = \prod_\sigma \alpha^{\sigma\tau} = \prod_{\sigma'} \alpha^{\sigma'} = \beta$, so by the preceding remarks, $\beta \in \mathbb{Q}$. However, α is an algebraic integer, and so every α^σ is an algebraic integer. This means that β is an algebraic integer, so $\beta \in \mathbb{Z}$. Since $|\beta| = \prod_\sigma |\alpha^\sigma| < 1$, we must have $\beta = 0$. This can only happen if $\alpha^\sigma = 0$ for some σ, i.e. $\alpha = 0$ as required. $\qquad\square$

12.4 Theorem. *Let G be a nonabelian simple group, and K_i a conjugacy class of G. If $|K_i|$ is a prime power, then $K_i = \{1\}$.*

Proof. Suppose that $|K_i| = p^a$ where p is a prime, $K_i \neq \{1\}$, and choose $g \in K_i$. Suppose that $\chi \neq 1_G$ is an irreducible character of G. Then $\ker \chi = 1$ since $\chi \neq 1_G$; and $Z(\chi)/\ker \chi = Z(G/\ker \chi) = 1$ implies that $Z(\chi) = 1$. If $p \nmid \chi(1)$, then $\chi(g) = 0$ by Theorem 12.3. By the column orthogonality relations, Corollary 5.7, we have

$$0 = \sum_{\chi \in \mathrm{Irr}_G} \chi(g)\overline{\chi(1)} = 1 + \sum_{\substack{\chi \in \mathrm{Irr}_G \\ p|\chi(1)}} \chi(g)\chi(1).$$

(The term 1 comes from the principal character 1_G.) Thus

$$-\frac{1}{p} = 1 + \sum_{\substack{\chi \in \mathrm{Irr}_G \\ p|\chi(1)}} \chi(g)\frac{\chi(1)}{p},$$

where the right side is an algebraic integer, but the left side is not, a contradiction. $\qquad\square$

12.5 Theorem (Burnside). *Let G be a group of order $p^a q^b$ where p, q are primes. Then G is solvable.*

Proof. If G has a nontrivial normal subgroup N, then by induction on $|G|$, both N and G/N are solvable, so G is also. So we may assume that G is simple. Furthermore, G is nonabelian simple; otherwise we are done.

We may suppose that $a > 1$. Let P be a Sylow p-subgroup of G. Then $Z(P) \neq 1$. Let $1 \neq g \in Z(P)$. Then $C_G(P) \supseteq P$, so the conjugacy class containing g has size $|g^G| = [G : C_G(P)] \,\big|\, [G : P] = q^b$, contrary to Theorem 12.4. $\qquad\square$