SIXTY-SIXTH ANNUAL
WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Saturday, December 3, 2005

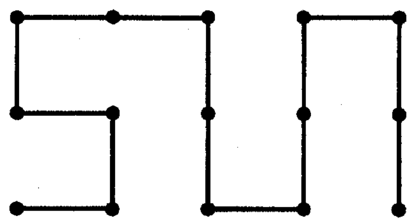
Examination A

Problem A1

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Problem A2

Let $S = \{(a,b) \mid a = 1,2,\ldots,n,\ b = 1,2,3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$, (iii) for each $p \in S$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)



Problem A3

Let $p(z)$ be a polynomial of degree n, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Problem A4

Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Problem A5

Evaluate $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

Problem A6

Let n be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are n randomly, independently and uniformly, chosen points on a circle. Consider the convex n-gon whose vertices are the P_i. What is the probability that at least one of the vertex angles of this polygon is acute?

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Examination B

Problem B1

Find a nonzero polynomial $P(x,y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a. (Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to v.)

Problem B2

Find all positive integers $n, k_1, \dots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem B3

Find all differentiable functions $f : (0,\infty) \to (0,\infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.

Problem B4

For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \dots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \le m$. Show that $f(m,n) = f(n,m)$.

Problem B5

Let $P(x_1,\ldots,x_n)$ denote a polynomial with real coefficients in the variables $x_1,\ldots,x_n$, and suppose that

(a) $$\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2}\right)P(x_1,\ldots,x_n)=0 \quad \text{(identically)}$$

and that

(b) $$x_1^2+\cdots+x_n^2 \text{ divides } P(x_1,\ldots,x_n).$$

Show that $P=0$ identically.

Problem B6

Let S_n denote the set of all permutations of the numbers 1, 2, … , n. For $\pi \in S_n$, let $\sigma(\pi)=1$ if π is an even permutation and $\sigma(\pi)=-1$ if π is an odd permutation. Also, let $\nu(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi\in S_n}\frac{\sigma(\pi)}{\nu(\pi)+1}=(-1)^{n+1}\frac{n}{n+1}.$$