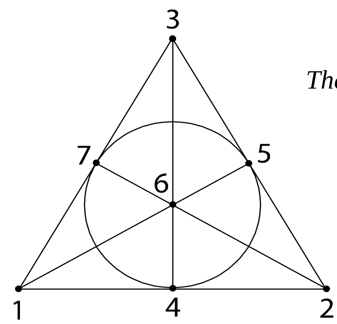
Projective Planes

A *projective plane* is an incidence system of points and lines satisfying the following axioms:

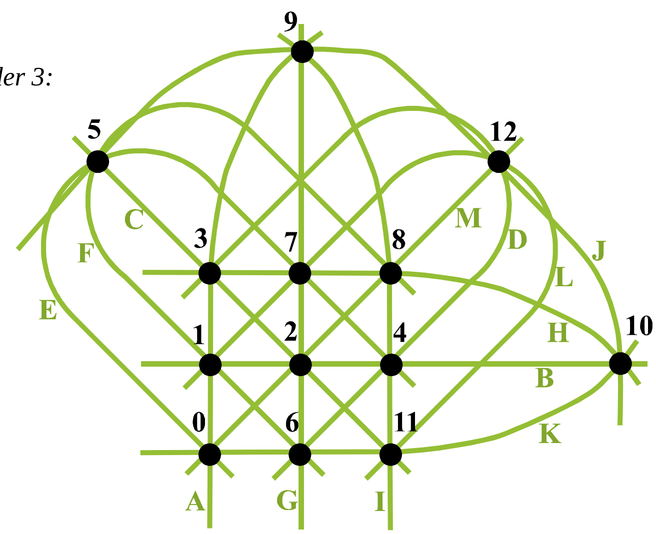
(P1) Any two distinct points are joined by exactly one line.
(P2) Any two distinct lines meet in exactly one point.
(P3) There exists a quadrangle: four points of which no three are collinear.

In class we will exhibit a projective plane with 57 points and 57 lines: the game of *SpotIt*®. (Actually the game is sold with only 55 lines; for the purposes of this class I have added the two missing lines.) The smallest projective plane, often called the Fano plane, has seven points and seven lines, as shown on the right.



The Fano Plane (of order 2):
7 points, 7 lines
3 points on each line
3 lines through each point

The Projective Plane of order 3:
13 points, 13 lines
4 points on each line
4 lines through each point



The second smallest projective plane, having thirteen points 0, 1, 2, …, 12 and thirteen lines A, B, C, …, M is shown on the left.

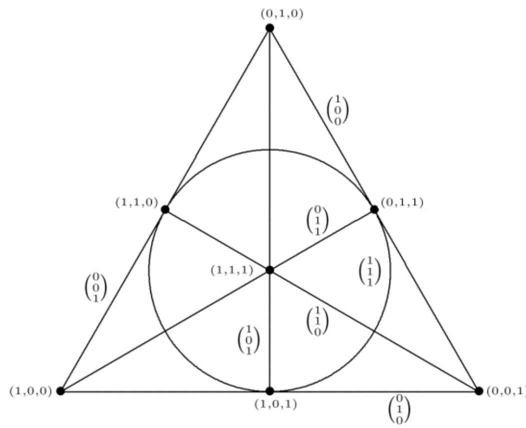
These two planes are coordinatized by the fields of order 2 and 3 respectively; this accounts for the term ‘order’ which we shall define shortly.

Given any field F, the *classical projective plane over* F is constructed from a 3-dimensional vector space $F^3 = \{(x,y,z) : x,y,z \in F\}$ as follows:

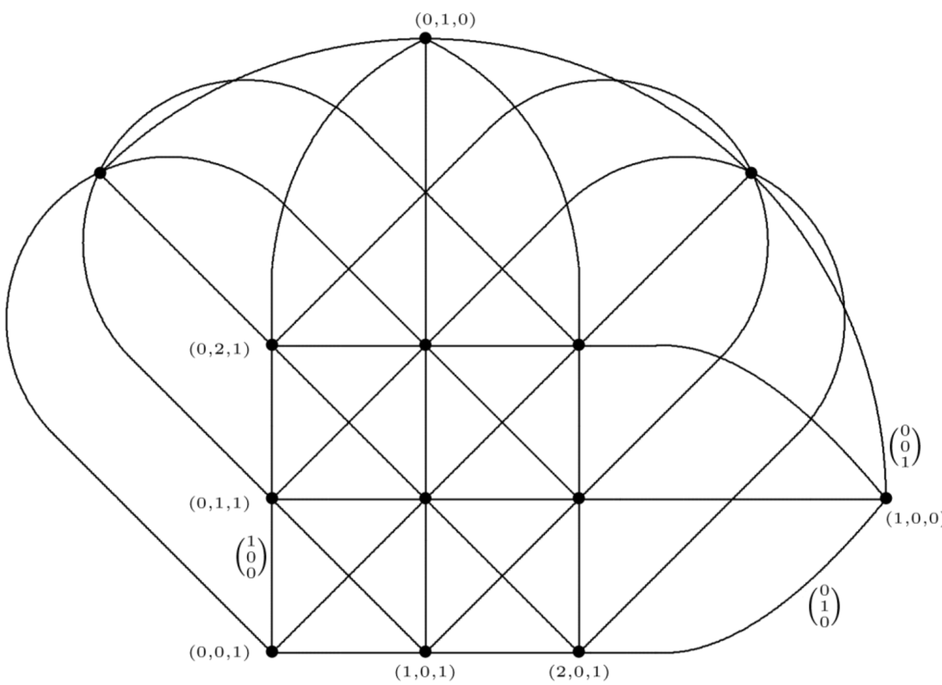
- ‘Points’ are one-dimensional subspaces $\langle(x,y,z)\rangle$. Here $(x,y,z) \in F^3$ is any *nonzero* vector; it spans a one-dimensional subspace $\langle(x,y,z)\rangle = \{\lambda(x,y,z) : \lambda \in F\}$. Recall that $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$.

- ‘Lines’ are two-dimensional subspaces $\{(x,y,z) : ax+by+cz=0\}$. Here $a,b,c \in F$ are not all zero. This line is represented by the column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$; but any nonzero scalar multiple $\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \\ \lambda c \end{pmatrix}$ represents the same line; so actually it is the span
$$\langle\begin{pmatrix} a \\ b \\ c \end{pmatrix}\rangle = \{\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} : \lambda \in F,\ \ \lambda \neq 0\}$$
that represents the line.
- Incidence is the usual containment between subspaces. The point $\langle(x,y,z)\rangle$ lies on the line $\langle\begin{pmatrix} a \\ b \\ c \end{pmatrix}\rangle$ iff $0 = (x,y,z)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = ax+by+cz.$

For example, the classical projective plane over $\mathbb{F}_2 = \{0,1\}$ is coordinatized as shown:



Here nonzero scalar multiples play essentially no role, since the only nonzero scalar is 1. A more representative example is the classical projective plane over $\mathbb{F}_3 = \{0,1,2\}$ which is coordinatized as shown:



I have not explicitly shown the coordinates for *all* points and lines, only a few; in class we filled in the missing coordinates for points and lines. Because nonzero scalars are 1 and 2 in this case, we have two equivalent ways to label each point and each line; for example $\langle(1,0,2)\rangle = \langle(2,0,1)\rangle$ and we have arbitrarily chosen to label the corresponding point as $(2,0,1)$.

In class we will prove:

> **Theorem**: Any two lines in a projective plane have the same number of points. This is the same as the number of lines through every point. If this number is $n+1$, then the plane has altogether n^2+n+1 points and the same number of lines.

The number n appearing in this result is called the *order* of the projective plane. What we *actually* show is that there is a bijection between the points on any line, and the points on any other line; also a bijection between the points on any line, and the lines through any point. This number (the order) may be finite or infinite.

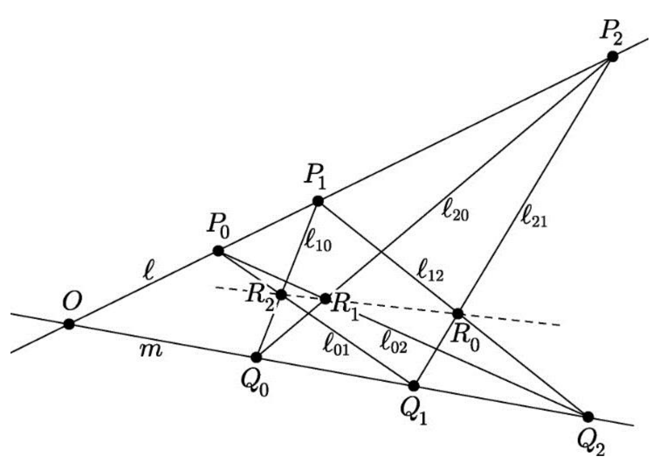
There is a strong connection between affine planes and projective planes. From a projective plane of order n, one constructs an affine plane of the same order by deleting any line and all $n+1$ points on that line. Conversely, given any affine plane of order n, one constructs a projective plane of the same order by adding a line 'at infinity' with $n+1$ points, one for each parallel class of lines in the affine plane. The same process works for planes of infinite order; in particular from the real affine plane (i.e. the Euclidean plane), we obtain the real projective plane and vice versa.

Projective (or affine planes) of order p^r exist for every prime p and integer $r \geq 1$. In particular since there exists a field of every prime-power order, this yields a classical plane of the corresponding order. Many non-classical finite planes (of prime-power order) are also known. However, it is not known if there exist any (projective or affine) planes of non-prime-power order. In particular, it is not known if there exists any plane of order 12.

Plane geometry from the projective viewpoint offers many advantages over the affine viewpoint; in particular, if we *dualize* a projective plane by reversing the roles of points and lines, we obtain another projective plane. This duality is not possible in the theory of affine planes since the dual of an affine plane is not an affine plane.

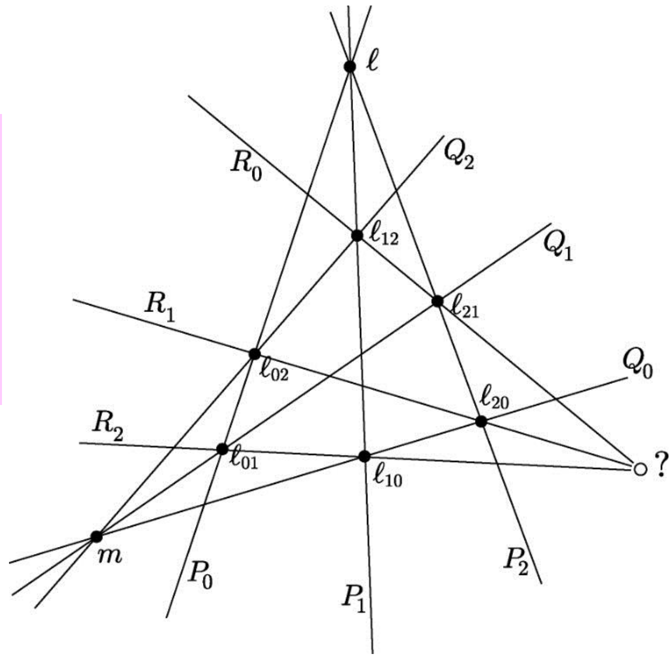
For example, consider Pappus' Theorem (valid in any classical plane):

> **Theorem (Pappus)**: Let ℓ and m be distinct lines. Let P_0, P_1, P_2 be distinct points of ℓ, and let Q_0, Q_1, Q_2 be distinct points of m. Consider the lines $\ell_{ij} = P_iQ_j$ for all i,j. For all i,j,k distinct, consider the point $R_k = \ell_{ij} \cap \ell_{ji}$. Then the points R_0, R_1, R_2 are collinear.



Dualizing gives the following corollary, valid also in any classical plane:

Theorem (Dual of Pappus' Theorem): Let ℓ and m be distinct points. Let P_0, P_1, P_2 be distinct lines through ℓ, and let Q_0, Q_1, Q_2 be distinct lines through m. Consider the points $\ell_{ij} = P_i \cap Q_j$ for all i, j. For all i, j, k distinct, consider the line $R_k = \ell_{ij}\ell_{ji}$. Then the lines R_0, R_1, R_2 are concurrent.



Both these theorems are valid in the affine plane; but in the affine setting, they would require two separate (lengthy) proofs. The economy of the projective viewpoint is that one proof suffices to prove both theorems. This is because the dual of any classical projective plane is again a classical projective plane. There are *many* other instances showing the superiority of the projective viewpoint.