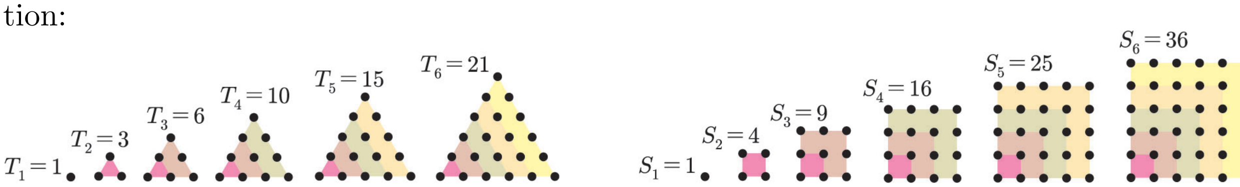
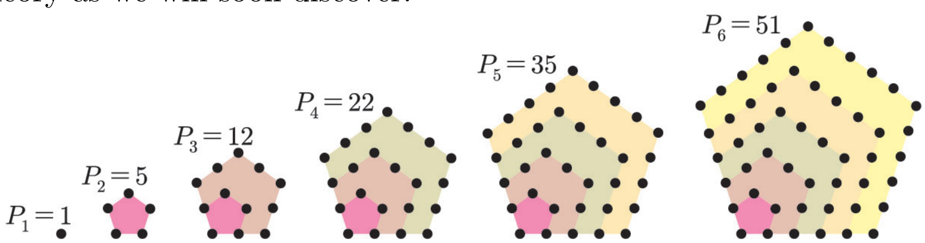
Pentagonal Numbers (Handout May, 2016)

The familiar sequences of triangular square numbers have a natural geometric interpretation:



This pattern extends to an entire sequence of sequences: the polygonal numbers. For larger k, the k-gonal numbers are less natural to motivate; but we find the sequence of pentagonal numbers worthy of special attention because of a surprising application to partition theory as we will soon discover:



Explicit formulas for triangular, square and pentagonal numbers, and the associated generating functions, are easily deduced:

$$T_n = \tfrac{1}{2}n(n+1) \qquad S_n = n^2 \qquad P_n = \tfrac{1}{2}n(3n-1)$$

$$\sum_{n=0}^{\infty} T_n x^n = \tfrac{x}{(1-x)^3} \qquad \sum_{n=0}^{\infty} S_n x^n = \tfrac{x(1+x)}{(1-x)^3} \qquad \sum_{n=0}^{\infty} P_n x^n = \tfrac{x(1+2x)}{(1-x)^3}$$

It is interesting to look at a table of values of T_n, S_n and P_n over a range of integer values of n, including some negative values of n:

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
T_n	28	21	15	10	6	3	1	0	0	1	3	6	10	15	21	28	36
S_n	64	49	36	25	16	9	4	1	0	1	4	9	16	25	36	49	64
P_n	100	77	57	40	26	15	7	2	0	1	5	12	22	35	51	70	92

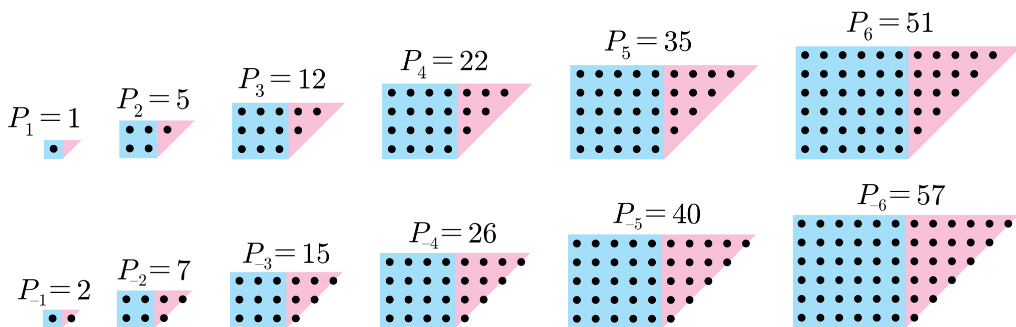
You should notice that the triangular and square numbers take the same values for $n \leqslant 0$ as for $n \geqslant 0$; this is explained by the relations

$$T_{-n-1} = T_n, \quad S_{-n} = S_n$$

which are easily verified algebraically. In the case of pentagonal numbers, however, we get new values

$$P_{-n} = \tfrac{1}{2}n(3n+1)$$

which are not attained by the formula $P_n = \frac{1}{2}n(3n-1)$ for $n > 0$. We may therefore consider a *pentagonal number* to be any number of the form $\frac{1}{2}n(3n \pm 1)$ for some $n \geqslant 0$; and it is now reasonable to rearrange the pentagonal numbers into a single sequence as $0, 1, 2, 5, 7, 12, 15, 22, \ldots$. Here we illustrate the first few pentagonal numbers graphically:



In the upper sequence ($n > 0$) each pentagon of side n is composed of a square of side n and a triangle of side $n-1$, giving

$$P_n = S_n + T_{n-1} = n^2 + \tfrac{1}{2}n(n-1) = \tfrac{1}{2}n(3n-1);$$

while in the lower sequence, each pentagon is formed by a square and a triangle, both of side n, yielding

$$P_{-n} = S_n + T_n = n^2 + \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(3n+1).$$

Although these pentagons appear less symmetrical than those in the original geometric picture, the depiction here as as Ferrers diagrams relates more directly to our study of partitions.

Partitions into Distinct Parts and Odd Parts

Recall the partition function $p(n)$, defined as the number of partitions of n, i.e. the number of ways to write n as a sum of positive integers, where the order of the terms does not matter. We have seen that the ordinary generating function for $p(n)$ is

$$(*) \qquad \sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} \sum_{r_k=0}^{\infty} x^{r_k k} = \prod_{k=0}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \cdots).$$

Recall the explanation for this formula: after collecting terms on the right, the coefficient of x^n is the number of solutions of $n = r_1 + 2r_2 + 3r_3 + \cdots$, i.e. the number of partitions of n in which there are exactly r_k parts of size k; and since the limits on k and the r_k's ensure that we count every partition of n exactly once, the coefficient of x^n on the right side is $p(n)$.

We now refine this counting problem by asking for $q(n)$, the number of partitions of n into *distinct* parts; and $p_o(n)$, the number of partitions of n into *odd* parts. For example, $q(8) = p_o(8) = 6$: the six partitions of 8 into *distinct* parts are

$$8, \quad 5{+}2{+}1, \quad 4{+}3{+}1, \quad 7{+}1, \quad 6{+}2, \quad 5{+}3$$

while the six partitions of 8 into *odd* parts are

$$7{+}1, \quad 5{+}3, \quad 5{+}1{+}1{+}1, \quad 3{+}3{+}1{+}1, \quad 3{+}1{+}1{+}1{+}1{+}1, \quad 1{+}1{+}1{+}1{+}1{+}1{+}1{+}1.$$

It is no coincidence that $q(8) = p_o(8)$; in general we have

Theorem 1. The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

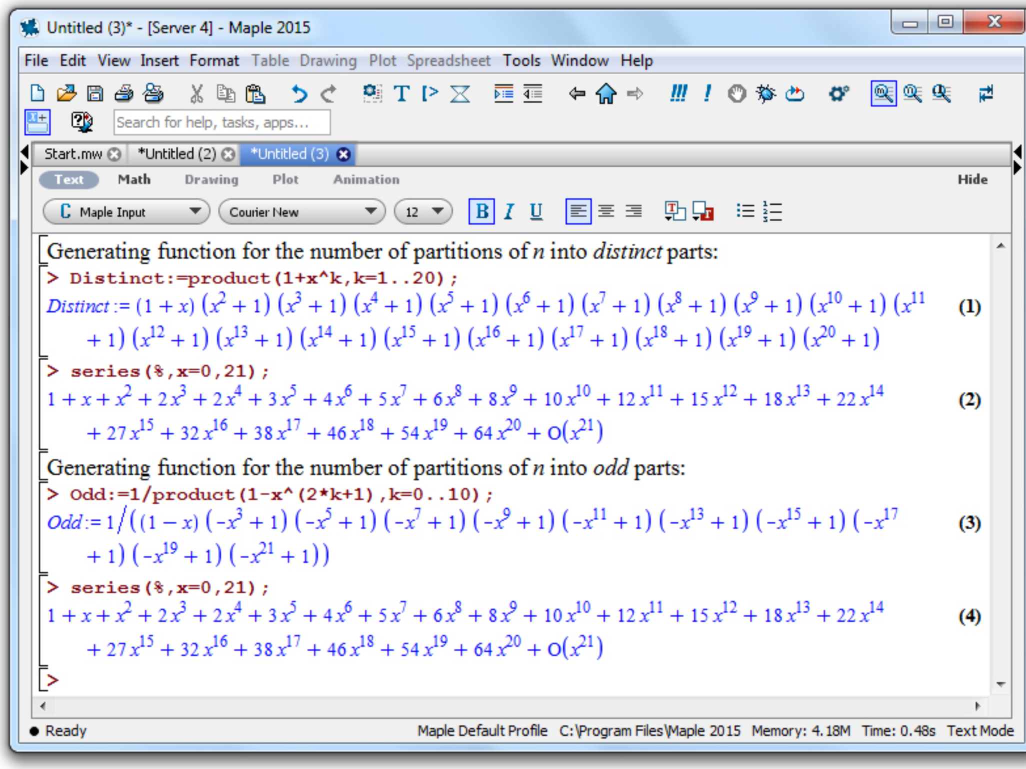
Proof. The generating function for $q(n)$ and $p_o(n)$ are found by modifying (*) to restrict the type of partitions considered. For $q(n)$ we restrict to those partitions of n having each term k appear at most once, i.e. $r_k \in \{0, 1\}$, which gives

$$\begin{aligned} Q(x) = \sum_{n=0}^{\infty} q(n)x^n \ &= \ \prod_{k=1}^{\infty}(1 + x^k) \ = \ (1 + x)(1 + x^2)(1 + x^3)\cdots \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots. \end{aligned}$$

For $p_o(n)$ we restrict to those partitions of n having only odd terms $k = 2j{+}1$ appear, each appearing any number of times $r_k \in \{0, 1, 2, 3, \ldots\}$, which gives

$$\begin{aligned} P_o(x) = \sum_{n=0}^{\infty} p_o(n)x^n \ &= \ \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j+1}} \ = \ \frac{1}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^9)\cdots} \\ &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \cdots. \end{aligned}$$

Using Maple we can verify that these two series agree to as many terms as desired, which certainly lends credibility to the statement we are trying to prove:



To prove Theorem 1, multiply numerator and denominator of the $P_o(x)$ expansion by $(1-x^2)(1-x^4)(1-x^6)\cdots$ to obtain

$$P_o(x) \;=\; \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)(1-x^{10})(1-x^{12})\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^6)(1-x^7)(1-x^8)\cdots}\,.$$

After factoring each factor $1-x^{2j} = (1-x^j)(1+x^j)$ in the numerator and cancelling factors with the denominator, we are left with

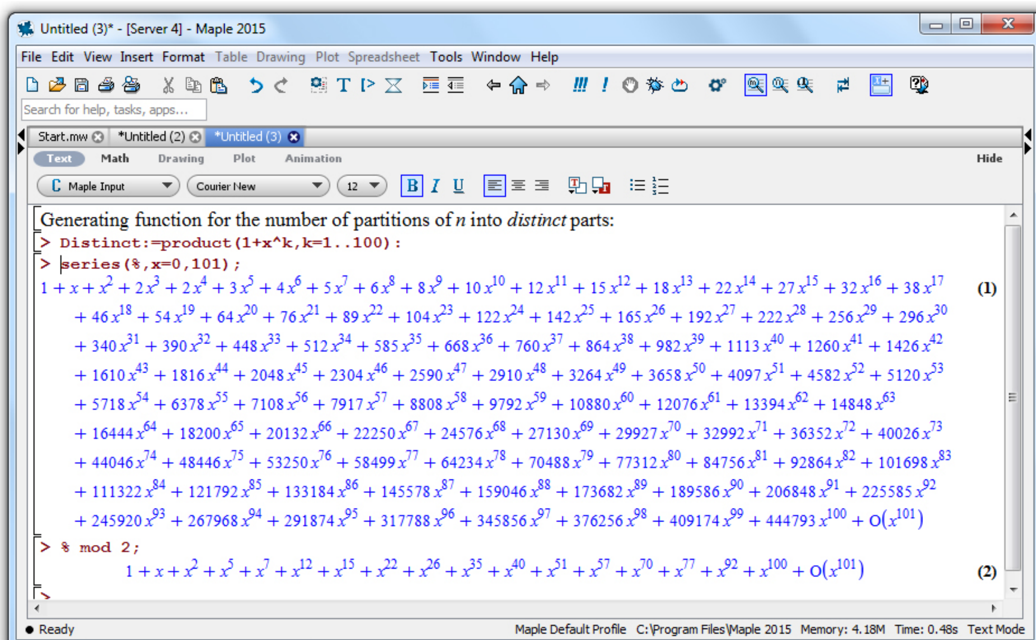
$$P_o(x) \;=\; (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots \;=\; Q(x).$$

Comparing the coefficient of x^n on each side gives $p_o(n) = q(n)$ as desired. $\square$

The preceding proof demonstrates the utility of generating functions; but we may be left to wonder why an algebraic proof should be needed to prove a strictly combinatorial fact. In fact a more combinatorial proof is possible. Such a proof would consist of an explicit bijection between the set of partitions of n into distinct parts, and the set of partitions of n into odd parts. This proof is a little less pretty (nothing as pretty as the use of conjugate partitions in giving a bijection between partitions of n into k parts, and partitions of n into parts of maximum size k). Rather than giving all the details, we only sketch the proof and give $n = 8$ as an example: Given a partition of n into distinct parts as $n = n_1 + n_2 + \cdots + n_k$, factor each $n_i = 2^{c_i} m_i$ where $c_i \geqslant 0$ and m_i is the largest odd divisor of n_i. Split n_i into 2^{c_i} parts of odd size m_i to obtain a partition of n into $\sum_{i=1}^{k} 2^{c_i}$ parts. The m_i's are not necessarily distinct (it is possible that $m_i = m_j$ for some $i \neq j$); nevertheless we obtain a one-to-one correspondence between partitions of n into distinct parts, and partitions of n into odd parts, essentially because the binary representation of every positive integer is unique (i.e. there is only one way to write a given positive integer as a sum of distinct powers of 2). Here we illustrate this bijection in the case $n = 8$:

$$\begin{aligned}
8 = 8{\cdot}1 \quad &\leftrightarrow \quad 1{+}1{+}1{+}1{+}1{+}1{+}1{+}1\\
7+1 \quad &\leftrightarrow \quad 7+1\\
6+2 = 2{\cdot}3 + 2{\cdot}1 \quad &\leftrightarrow \quad (3{+}3)+(1{+}1)\\
5+3 \quad &\leftrightarrow \quad 5+3\\
5+2+1 = 5+2{\cdot}1+1 \quad &\leftrightarrow \quad (5)+(1{+}1)+(1)\\
4+3+1 = 4{\cdot}1+3+1 \quad &\leftrightarrow \quad (1{+}1{+}1{+}1)+(3)+(1{+}1)
\end{aligned}$$

Finally, for the promised connection to pentagonal numbers, we look at the pattern of even and odd coefficients in the generating function $Q(x) = P_o(x)$. For this we simply reduce modulo 2:



What you should observe is that the exponents that appear in the latter sum are precisely the pentagonal numbers; i.e. $q(n) = p_o(n)$ is odd if n is a pentagonal number, and even otherwise. The explanation for this observation is the following: Denote by $q_e(n)$ and $q_o(n)$

the number of partitions of n into an even number of distinct parts, and an odd number of distinct parts, respectively, so that

$$q_e(n) + q_o(n) = q(n).$$

Theorem 2. If n is not a pentagonal number, then $q_e(n) = q_o(n)$ and so $q(n) = 2q_o(n)$ which is even. If n is a pentagonal number, say $n = P_j$, then $q_e(n) = q_o(n) + (-1)^j$ and so $q(n) = 2q_o(n) + (-1)^j$ which is odd.

From our enumeration of the six partitions of 8 into distinct parts, we have seen that $q_e(8) = q_o(8) = 3$ as predicted by Theorem 2 since 8 is not a pentagonal number. In the case $n = 7 = P_{-2}$ we have $q_e(7) = 3$ partitions into an even number of distinct parts:

6+1, 5+2, 4+3;

and $q_o(7) = 2$ partitions into an odd number of distinct parts:

7, 4+2+1,

as predicted by Theorem‘2. In the case $n = 12 = P_3$, we have $q_e(12) = 7$ partitions into an even number of distinct parts:

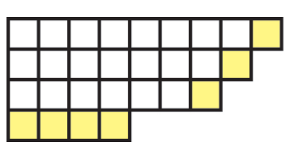
11+1, 10+2, 9+3, 8+4, 7+5, 6+3+2+1, 5+4+2+1;

and $q_o(12) = 8$ partitions into an odd number of distinct parts:

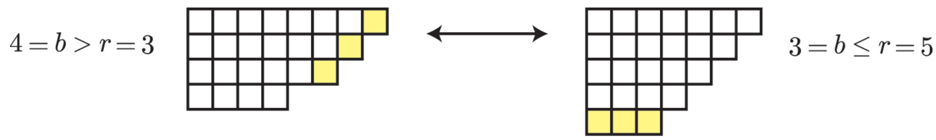
12, 9+2+1, 8+3+1, 7+4+1, 7+3+2, 6+5+1, 6+4+2, 5+4+3,

once again as predicted by Theorem 2.

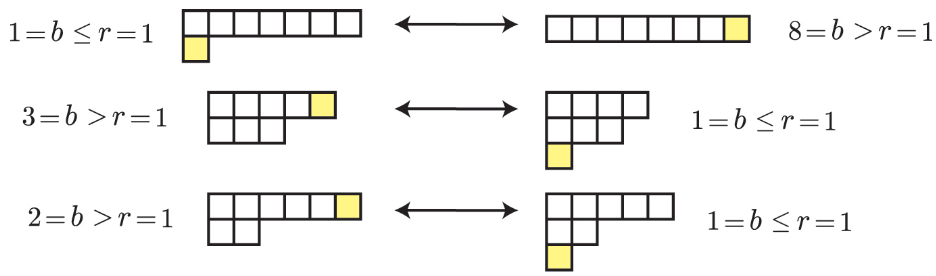
The key to proving Theorem 2 is the following almost-bijective correspondence between partitions of n with an even number of parts, and partitions of n with an odd number of parts. Given a Ferrers diagram for a partition, denote by b the length of the bottom row (i.e. the size of the smallest part in the partition) and let r number of cells on the rightmost 45° line. In the following example, we have $b = 4$ and $r = 3$:



If $b \leqslant r$, move the bottom row to the rightmost 45° line; but if $b > r$, move the rightmost 45° line down to the bottom. Here is one corresponding pair of partitions for $n = 25$:



and here is the complete correspondence for $n = 8$:



We obtain a well-defined bijection between partitions with an even number of distinct parts, and partitions with an odd number of distinct parts, *except* when n is a pentagonal number. If $n = P_j$ where $j > 0$, then the correspondence fails just for the pentagonal Ferrers diagram with j rows having $b = r = j$; whereas if $n = P_{-j}$ where $j > 0$, then the correspondence fails just for the pentagonal Ferrers diagram with j rows having $b = r + 1$ and $r = j$. Consider what happens in the cases $n = P_4 = 22$ and $n = P_{-4} = 26$ as shown:

When n is not a pentagonal number, no such pentagonal Ferrers diagram exists, and we obtain a well-defined bijection between partitions with an even number of distinct parts, and partitions with an odd number of distinct parts, giving $q_e(n) = q_o(n)$. For a pentagonal number $n = P_j$, there is just one left-over partition not covered by the bijection, and it has j parts, so $q_e(n) = q_o(n) + (-1)^j$. This proves the theorem. □

Just as

$$Q(x) = \sum_{n=0}^{\infty} q(n)x^n = \sum_{n=0}^{\infty} \big(q_e(n) + q_o(n)\big)x^n = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots,$$

we see that

$$\begin{aligned}(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots &= \sum_{n=0}^{\infty}\big(q_e(n)-q_o(n)\big)x^n \\ &= \sum_{j=0}^{\infty}(-1)^j x^{P_j} \\ &= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots\end{aligned}$$

in which the only surviving terms are those whose exponents are pentagonal numbers! The reason is that positive terms x^n in the expansion of the latter product, correspond to partitions of n into an even number of distinct parts; whereas negative terms $-x^n$ correspond to partitions of n into an odd number of distinct parts. Noting that the latter product is the reciprocal of

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots},$$

we obtain the curious relation

$$(1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots)\sum_{n=0}^{\infty} p(n)x^n = 1.$$

Comparing terms on both sides gives a recurrence formula for the partition function:

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\cdots$$

where we stop as soon as the argument becomes negative.