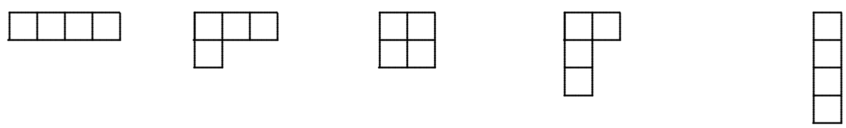
UNIVERSITY OF WYOMING
Math 3700—Spring 2023
Combinatorics
Department of Mathematics

The Partition Function

The partition function $p(n)$ expresses the number of ways of partitioning n identical objects into nonempty piles, where the order of the piles does not matter. For example, $p(4) = 5$ since we have

$$4 \quad = \quad 3{+}1 \quad = \quad 2{+}2 \quad = \quad 2{+}1{+}1 \quad = \quad 1{+}1{+}1{+}1.$$

Each partition is denoted as a tuple in which the sizes of the parts ('piles') are listed in weakly decreasing order; for example the partition $4 = 2{+}1{+}1$ is denoted by $(2, 1, 1) \vdash 4$ where the symbol '$\vdash$' means 'is a partition of'. We also denote each partition graphically by a **Ferrers diagram** (or **Young diagram**) whose rows indicate the parts of the partition. For example, here are the five partitions of 4, together with their Ferrers diagrams:



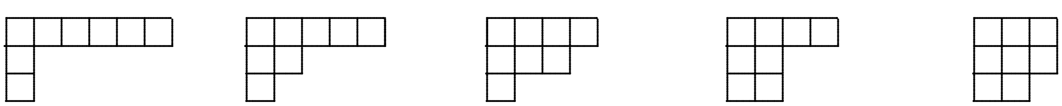
4   3+1   2+2   2+1+1   1+1+1+1

Here are the first few values of the partition function:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$p(n)$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231

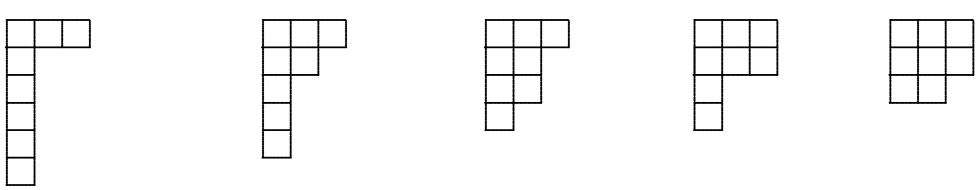
In slightly different language, $p(n)$ is the number of partitions of n into nonempty parts. It should be clear why we require the parts to be nonempty: without this requirement, we could have an unlimited number of empty parts (e.g. $4{+}0 = 4{+}0{+}0 = 4{+}0{+}0{+}0 =$ etc.) with the resulting number of 'partitions' being infinite, whcih we clearly want to avoid.

Refining our count, we have $p(n) = \sum_{k=1}^{n} p_k(n)$ where $p_k(n)$ is the number of partitions of n into k nonempty parts. Thus for example $p_3(8) = 5$:



$(6, 1, 1)$ $(5, 2, 1)$ $(4, 3, 1)$ $(4, 2, 2)$ $(3, 3, 2)$

Rather than limiting the number of parts, we may choose to limit the size of each part. For example there are exactly five partitions of 8 into nonempty parts of maximum size 3:



$(3, 1, 1, 1, 1, 1)$ $(3, 2, 1, 1, 1)$ $(3, 2, 2, 1)$ $(3, 3, 1, 1)$ $(3, 3, 2)$

Note that the preceding list of Ferrers diagrams comes from the previous list, by reflection across the $-45°$ line through the upper left corner. This operation is called **conjugation**; for example, the conjugate of the partition $(6, 1, 1)$ is the partition $(3, 1, 1, 1, 1, 1)$. Note that the partition $(3, 3, 2)$ is conjugate to itself; it is **self-conjugate**. Evidently, conjugation establishes a one-to-one correspondence between partitions of n having k parts, and partitions of n having largest part k. In each case, the number of partitions is $p_k(n)$. Note that the number of partitions of n having parts of size $\leqslant k$ is not $p_k(n)$, but rather $p_1(n)+p_2(n)+\cdots+p_k(n)$.

In summary, a **partition** of an integer n with k parts is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ satisfying $\lambda_1+\lambda_2+\cdots+\lambda_k = n$. If these conditions are satisfied, we write $\lambda \vdash n$; and we call $\lambda_1, \lambda_2, \ldots, \lambda_k$ the **parts** of the partition.

Theorem 1. The generating function for the partition function $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k}\,.$$

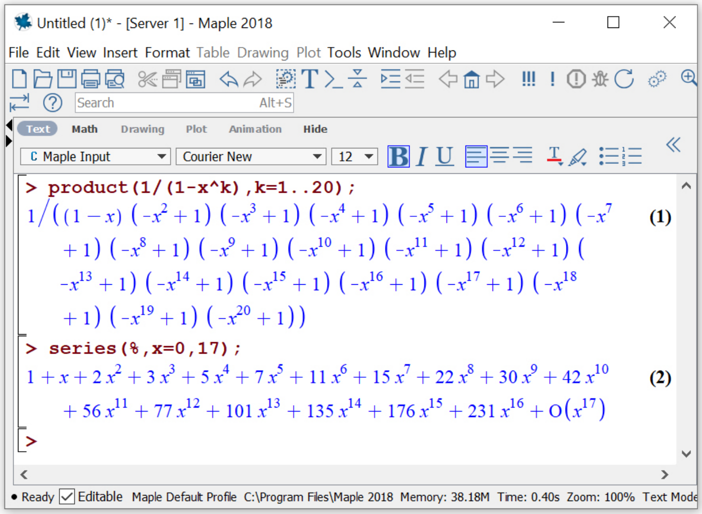
Proof.
$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + x^{3k} + x^{4k} + \cdots)$$
$$= (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots) \times \cdots.$$

A general term in this expansion has the form $x^{r_1+2r_2+3r_3+4r_4+\cdots}$ where $r_1, r_2, r_3, r_4, \ldots$ are non-negative integers. Now we collect terms. The coefficient of x^n in the expansion is of course the number of tuples of non-negative integers $(r_1, r_2, r_3, r_4, \ldots)$ satisfying

$$(*) \qquad r_1+2r_2+3r_3+4r_4+\cdots = n.$$

Evidently every solution of (*) has only finitely many positive r_i's, beyond which all the remaining r_i's must be zero. Moreover every solution of (*) corresponds to a partition of n in which we have r_1 parts of size 1, r_2 parts of size 2, r_3 parts of size 3, etc. So the number of solutions of (*) is exactly $p(n)$, the number of partitions of n. This gives the result. □

Typical symbolic computation engines will not be able to store the infinite product $\prod_{k=1}^{\infty} \frac{1}{1-x^k}$ directly. Instead, a finite product $\prod_{k=1}^{m} \frac{1}{1-x^k}$ may be used with m sufficiently large. Indeed, the series expansion of the full generating function in all terms up to degree m. So by taking $m \geqslant n$ and reading coefficients of the power series expansion, we may correctly compute $p(0), p(1), p(2), \ldots, p(n)$. The following Maple session evaluates the values of $p(n)$ given in our earlier table of values:



Theorem 2. Fix a positive integer k.

(a) The generating function for $p_k(n)$, the number of partitions of n with k parts (or largest part k) is

$$\sum_{n=1}^{\infty} p_k(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}\,.$$

(b) The generating function for $p_1(n)+p_2(n)+\cdots+p_k(n)$, the number of partitions of n with *at most* k parts (or parts of size *at most* k) is

$$\sum_{n=1}^{\infty} \big(p_1(n)+p_2(n)+\cdots+p_k(n)\big)x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}\,.$$

Proof. We first prove (b). We may interpret $p_1(n)+p_2(n)+\cdots+p_k(n)$ as the number of partitions of n into parts of size at most k (since by conjugation, we know that this is the same as the number of partitions of n into at most k parts). Now

$$\tfrac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)\times\cdots\times(1+x^k+x^{2k}+x^{3k}+\cdots).$$

A typical term in the expansion of this product has the form $x^{r_1+2r_2+3r_3+\cdots+kr_k}$ where $r_1, r_2, \ldots, r_k$ are non-negative integers satisfying

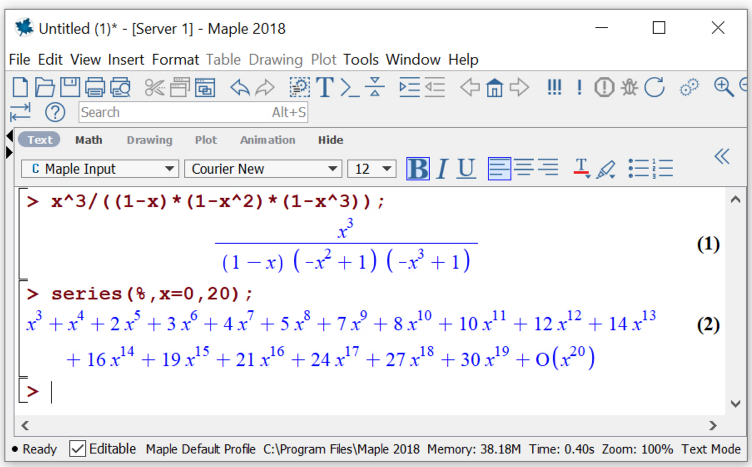
$$(\dagger) \qquad r_1+2r_2+3r_3+\cdots+kr_k = n.$$

Again, every solution of $(\dagger)$ corresponds to a partition of n into parts of size at most k (by taking r_1 parts of size 1, r_2 parts of size 2, …, r_k parts of size k). So the number of solutions of $(\dagger)$ is $p_1(n)+p_2(n)+\cdots+p_k(n)$, the number of partitions of n into parts. This gives (b).

For (a), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} p_k(n)x^n &= \sum_{n=1}^{\infty}\big(p_1(n)+p_2(n)+\cdots+p_k(n)\big)x^n - \sum_{n=1}^{\infty}\big(p_1(n)+p_2(n)+\cdots+p_{k-1}(n)\big)x^n \\
&= \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})(1-x^k)} - \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})} \\
&= \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})}\left[\frac{1}{1-x^k} - 1\right] \\
&= \frac{1}{(1-x)(1-x^2)\cdots(1-x^{k-1})}\cdot\frac{x^k}{1-x^k}. \qquad \square
\end{aligned}
$$

Previously we determined $p_3(8) = 5$ by explicitly enumerating partitions of 8 with 3 parts (also partitions of 8 with largest part 3). Here is a Maple session in which $p_3(8) = 5$ can be read from the coefficient of x^8 using Theorem 2(a):



Note that we my recover Theorem 1 from Theorem 2(b) by letting $k \to \infty$. In the limit we have the generating function $\lim_{k\to\infty} \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)} = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$. The coefficient of x^n in the series expansion of the infinite product is $p_1(n)+p_2(n)+p_3(n)+\cdots = p(n)$.