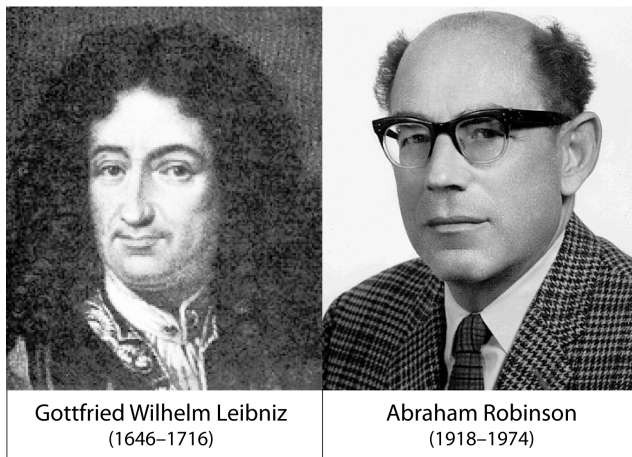


## Ultraproducts and Hyperreal Numbers

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Notions of infinite and infinitesimal numbers have been around since the earliest days of calculus; in particular, Leibniz found them a great inspiration in his co-invention of calculus. Later generations of analysts, including Weierstrass, frowned upon such notions, pointing out the lack of rigor in many of their arguments. Yet even Cauchy, who was largely known for his efforts to ‘clean up’ analysis and put it on a rigorous foundation, was prone to arguments motivated by infinitesimals. It was not until the twentieth century that the use of infinite and infinitesimal quantities in analysis was given a rigorous mathematical foundation by Robinson. Today many mathematicians view this ‘nonstandard approach’ to analysis as fringe science, and frown upon it. Others find nonstandard analysis to be of great pedagogical value, and an invaluable source of inspiration in mathematical research. In any case, tradition has so firmly embedded the ‘standard’ approach to calculus in our undergraduate curriculum, that the nonstandard alternative is not likely to play a key role anytime soon; and for students who struggle with the very foundations of the real number system, introducing the hyperreals is likely to add to the confusion. Yet the professional mathematician will still benefit from the enlightened position that neither the real number system  $\mathbb{R}$ , nor the hyperreal number system  $\widehat{\mathbb{R}}$  is physical reality; both  $\mathbb{R}$  and  $\widehat{\mathbb{R}}$  are idealized descriptions of reality, each with its own strengths and limitations.



Gottfried Wilhelm Leibniz  
(1646–1716)

Abraham Robinson  
(1918–1974)

We proceed here to extend the real number system  $\mathbb{R}$  to the field  $\widehat{\mathbb{R}}$  of hyperreal numbers, including both ‘infinitesimals’ and ‘infinite quantities’, while retaining well-defined field operations of addition, subtraction, multiplication and division, without running into trouble with indeterminate forms. To be precise, we construct a field extension  $\widehat{\mathbb{R}} \supset \mathbb{R}$  of **hyperreals** with the following properties:

- (N1)  $\widehat{\mathbb{R}}$  is an **ordered field**. Thus  $\widehat{\mathbb{R}}$  has well-defined field operations of addition, subtraction, multiplication and division (only division by zero is of course not allowed). Elements of the subfield  $\mathbb{R}$  are referred to as **standard reals**. The

algebraic operations of addition and multiplication satisfy the usual laws of commutativity, associativity and distributivity. Moreover  $\widehat{\mathbb{R}}$  is totally ordered; and this order relation is compatible with the algebraic operations; so for example if  $a, b, c \in \widehat{\mathbb{R}}$  satisfy  $a < b$  and  $c > 0$ , then  $ac < bc$ ; etc.

We have seen that (N1) is expressible in the first order theory of ordered rings with two constant symbols ‘0’, ‘1’; two binary operations ‘+’, ‘·’; and one binary relation symbol ‘<’. (Here we refer simply to the first-order theory of ordered *rings*, rather than ordered fields. This is because we have no formal symbol for division in our language; division is recovered from multiplication using the field axiom  $(\forall x)(\neg(x = 0) \rightarrow (\exists y)(xy = 1))$ .) The biggest difference between  $\widehat{\mathbb{R}}$  and  $\mathbb{R}$  is the existence of infinite and infinitesimal elements:

- (N2)  $\widehat{\mathbb{R}}$  contains ‘infinitesimals’, i.e. there exists  $\varepsilon \in \widehat{\mathbb{R}}$  such that  $0 < \varepsilon < \frac{1}{n}$  for every positive integer  $n$ . Equivalently (by considering  $\alpha = \frac{1}{\varepsilon}$ ),  $\widehat{\mathbb{R}}$  contains ‘infinite quantities’, i.e. elements larger than every positive integer.

In addition,  $\widehat{\mathbb{R}}$  satisfies all the same statements (of ordered ring theory) that  $\mathbb{R}$  does:

- (N3) Let  $\phi$  be a statement in the first-order language of ordered rings. Then  $\widehat{\mathbb{R}} \models \phi$  iff  $\mathbb{R} \models \phi$ .

As examples of (N3) we have:

- *Every positive element of  $\widehat{\mathbb{R}}$  has a square root:*

$$\widehat{\mathbb{R}} \models (\forall x)((x > 0) \rightarrow (\exists y)(x = y^2))$$

From this it follows that every positive element has exactly two square roots, one positive and the other negative. Here we abbreviate  $y \cdot y = y^2$ ,  $y \cdot y \cdot y = y^3$ , etc.

- *Every element of  $\widehat{\mathbb{R}}$  has a unique cube root:*

$$\widehat{\mathbb{R}} \models (\forall x)(\exists y)(x^3 = y) \wedge (\forall u)(\forall v)((u^3 = v^3) \leftrightarrow (u = v))$$

More generally, *every polynomial of degree three has a root:*

$$\widehat{\mathbb{R}} \models (\forall a)(\forall b)(\forall c)(\exists x)(x^3 + ax^2 + bx + c = 0)$$

Even more generally, every polynomial of odd degree has a root.

- *Polynomials satisfy the Intermediate Value Theorem.* For example in the case of polynomials of degree four, this says

$$\begin{aligned} \widehat{\mathbb{R}} \models & (\forall c_0)(\forall c_1)(\forall c_2)(\forall c_3)(\forall a)(\forall b) \\ & (((a^4 + c_3a^3 + c_2a^2 + c_1a + c_0 < 0) \wedge (0 < b^4 + c_3b^3 + c_2b^2 + c_1b + c_0)) \\ & \rightarrow (\exists x)((a < x) \wedge (x < b) \wedge (x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 = 0))) \end{aligned}$$

If one only wants to construct an ordered field containing infinitesimals, i.e. satisfying just (N1) and (N2), then a much simpler construction (avoiding the ultraproducts of

Sections 1–3) suffices. Start with the field  $\mathbb{R}[\alpha]$  consisting of all polynomials in the symbol ‘ $\alpha$ ’ with real coefficients. This is an ordered ring, where the order relation between two polynomials is found by comparing coefficients of highest degree. Thus ‘ $\alpha$ ’ satisfies all the expectations of an infinite quantity (and we may substitute the symbol ‘ $\infty$ ’ for  $\alpha$  if desired). The quotient field of  $\mathbb{R}[\alpha]$  gives an ordered field  $\mathbb{R}(\alpha)$  with infinite and infinitesimal elements. However, this field fails (N3); for example  $\alpha$  has no cube root in  $\mathbb{R}(\alpha)$ .

Also, if one wants only to show the existence of a field satisfying (N1)–(N3), then the Compactness Theorem suffices for this purpose. As we have seen, (N1) can be expressed using the axioms for ordered fields. These, and all the assertions given in (N3), can be expressed in the first-order theory of ordered rings. Now add an extra constant symbol ‘ $\alpha$ ’ to the language, plus an infinite list of axioms  $\alpha > 1$ ,  $\alpha > 2$ ,  $\alpha > 3$ , etc. Altogether we now have an infinite list of axioms; and any finite subset of this list can be realized in  $\mathbb{R}$  (by interpreting  $\alpha$  as a sufficiently large real constant) so by the Compactness Theorem, the entire (infinite) list of axioms has a model  $\widehat{\mathbb{R}}$ . While this argument suffices as a proof of *existence* of a field  $\widehat{\mathbb{R}}$  with the required properties, the construction of Sections 1–3 gives a somewhat more explicit construction of  $\widehat{\mathbb{R}}$ , leading to better understanding of its properties.

## 1. Filters

Let  $X$  be an arbitrary set. A **filter** on  $X$  is a collection  $\mathfrak{F}$  of subsets of  $X$  such that

- (F1)  $\emptyset \notin \mathfrak{F}$ ,  $X \in \mathfrak{F}$ .
- (F2) For all  $U, V \in \mathfrak{F}$  we have  $U \cap V \in \mathfrak{F}$ .
- (F3) Whenever  $U \in \mathfrak{F}$  and  $U \subseteq V \subseteq X$ , we have  $V \in \mathfrak{F}$ .

**1.1 Example: Principal Filters** For example let  $S \subseteq X$  be any nonempty subset:  $\emptyset \subset S \subseteq X$ . The collection of all supersets of  $S$  gives

$$\mathfrak{F}_S = \{U \subseteq X : U \supseteq S\}.$$

It is easy to verify that this is in fact a filter. Any example of this type is called a **principal filter**.

**1.2 Example: Finite Complement Filters** Suppose that  $X$  is an infinite set, and let  $\mathfrak{F}$  be the collection of all complements of finite subsets of  $X$ :

$$\mathfrak{F} = \{U \subseteq X : |X \setminus U| < \infty\}.$$

It is easy to verify that  $\mathfrak{F}$  is a filter on  $X$ , and that  $\mathfrak{F}$  is non-principal.

**1.3 Example: ‘Almost-Everywhere’ Sets** Consider the unit interval  $X = [0, 1]$  in the real line. Let  $\mathfrak{F}$  be the collection of all subsets  $U \subseteq [0, 1]$  of Lebesgue measure 1,

i.e.  $\lambda(U) = 1$ . For example if  $U \subseteq [0, 1]$  is the complement of some countable set then  $\lambda(U) = 1$  and so  $U \in \mathfrak{F}$ . But there exist subsets  $U \in \mathfrak{F}$  not of this form: for example if  $C \subset [0, 1]$  is the Cantor set then  $C$  is uncountable and yet  $\lambda(C) = 0$  because for every  $n \geq 1$ ,  $C$  may be covered by  $2^n$  intervals each of width  $3^{-n}$ ; thus  $U = [0, 1] \setminus C \in \mathfrak{F}$ .

If  $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$  where  $\mathfrak{F}_1, \mathfrak{F}_2$  are filters on  $X$ , we say that  $\mathfrak{F}_1$  **refines**  $\mathfrak{F}_2$ . Thus for example the filter of Example 1.3 strictly refines that of Example 1.2.

In view of Example 1.3, the reader should think of sets  $U \in \mathfrak{F}$  as ‘large subsets of  $X$ ’, those subsets that cover ‘almost all’ points of  $X$ . Recall that one often identifies functions  $f : X \rightarrow Y$  that agree except on a set of measure zero; for example to define  $L^1([0, 1])$  we start with Lebesgue-measurable functions  $f : [0, 1] \rightarrow \mathbb{R}$  such that  $\int_{[0,1]} |f(t)| dt < \infty$ ; we then identify any two such functions if they differ at most on a subset of  $[0, 1]$  of Lebesgue measure zero. This idea generalizes easily as follows: Let  $\mathfrak{F}$  be a filter on  $X$  and let  $Y$  be any set. If  $f, g : X \rightarrow Y$  are any two functions, write  $f \sim g$  (and say  $f$  is *equivalent* to  $g$ ) if  $f$  and  $g$  agree on some set  $U \in \mathfrak{F}$ , i.e.  $\{x \in X : f(x) = g(x)\} \in \mathfrak{F}$ . It is straightforward to check that this gives an equivalence relation on  $Y^X$ , the set of all functions  $X \rightarrow Y$ . For example if  $f \sim g \sim h$ , say  $f$  agrees with  $g$  on  $U \in \mathfrak{F}$ , and  $g$  agrees with  $h$  on  $V \in \mathfrak{F}$ , then  $f$  agrees with  $h$  on  $U \cap V \in \mathfrak{F}$ ; this proves transitivity of the relation ‘ $\sim$ ’.

Let  $\mathfrak{S}$  be any collection of subsets of  $X$ . We say  $\mathfrak{S}$  has the **finite intersection property** if every finite subset of  $\mathfrak{S}$  has nontrivial intersection, i.e.

$$S_1, S_2, \dots, S_n \in \mathfrak{S} \quad \text{implies} \quad S_1 \cap S_2 \cap \dots \cap S_n \neq \emptyset.$$

**Proposition 1.4.** Let  $\mathfrak{S}$  be a collection of subsets of  $X$ . Then  $\mathfrak{S}$  extends to a filter  $\mathfrak{F}$  on  $X$ , iff  $\mathfrak{S}$  has the finite intersection property. In this case the unique minimal filter containing  $\mathfrak{S}$  is given by

$$\mathfrak{F}_{\mathfrak{S}} = \{U \subseteq X : U \supseteq S_1 \cap S_2 \cap \dots \cap S_n \text{ for some } S_1, S_2, \dots, S_n \in \mathfrak{S}\}.$$

The filter  $\mathfrak{F}_{\mathfrak{S}}$  of Proposition 1.4 is called the **filter generated by  $\mathfrak{S}$** . Note that if  $\mathfrak{S} = \{S\}$  is a singleton, this gives the construction of principal filters of Example 1.1.

*Proof of Proposition 1.4.* It is clear from (F1) and (F2) that  $\mathfrak{S}$  cannot extend to a filter unless it has the finite intersection property. Assuming  $\mathfrak{S}$  has this property, it is also clear that any filter  $\mathfrak{F} \supseteq \mathfrak{S}$  must satisfy  $\mathfrak{F} \supseteq \mathfrak{F}_{\mathfrak{S}}$ ; and that  $\mathfrak{F}_{\mathfrak{S}}$  is itself a filter.  $\square$

## 2. Ultrafilters

We observe that no two members of a filter can be disjoint: if  $U, V \in \mathfrak{F}$  where  $\mathfrak{F}$  is a filter on  $X$ , then  $U \cap V \in \mathfrak{F}$  and so  $U \cap V \neq \emptyset$ . In particular if  $A \subseteq X$  then the subsets  $A$  and  $X \setminus A$  cannot both be in  $\mathfrak{F}$ . If  $\mathfrak{F}$  is a filter on  $X$  such that for every set  $A \subseteq X$  we have either  $A \in \mathfrak{F}$  or  $X \setminus A \in \mathfrak{F}$ , we say  $\mathfrak{F}$  is an **ultrafilter**. If  $\mathfrak{F}$  is an ultrafilter, it is clear that no filter properly refines  $\mathfrak{F}$ . In fact if one accepts the Axiom of Choice (and we usually do! otherwise the study of topology would be limited to rather trivial examples) then every filter can be refined to an ultrafilter.

**Theorem 2.1.** Let  $\mathfrak{F}_0$  be a filter on  $X$ . Then there exists an ultrafilter  $\mathfrak{U}$  on  $X$  such that  $\mathfrak{U} \supseteq \mathfrak{F}_0$ .

*Proof.* We use Zorn's Lemma, applied to the collection  $\mathfrak{S}$  of all filters  $\mathfrak{F}$  on  $X$  which refine  $\mathfrak{F}_0$ , partially ordered by refinement (i.e. inclusion). Since  $\mathfrak{F}_0 \in \mathfrak{S}$ , this collection is nonempty. If  $\mathcal{C}$  is any chain in  $\mathfrak{S}$ , then clearly  $\bigcup \mathcal{C}$  is an upper bound for  $\mathcal{C}$ . Verifying that  $\bigcup \mathcal{C} \in \mathfrak{S}$  is straightforward: for example  $\emptyset \notin \bigcup \mathcal{C}$  since every member of  $\mathcal{C}$  is itself a filter. If  $U, U' \in \bigcup \mathcal{C}$  then  $U \in \mathfrak{C}$  and  $U' \in \mathfrak{C}'$  for some  $\mathfrak{C}, \mathfrak{C}' \in \mathcal{C}$ ; but since  $\mathcal{C}$  is a chain, we may assume without loss of generality that  $\mathfrak{C} \subseteq \mathfrak{C}'$ , whence  $U \cap U' \in \mathfrak{C}' \subseteq \bigcup \mathcal{C}$ . The remaining properties required of  $\bigcup \mathcal{C}$  to be a filter, are verified similarly.

By Zorn's Lemma, we may take  $\mathfrak{U} \in \mathfrak{S}$  to be maximal. Let  $A \subseteq X$  and suppose  $A \notin \mathfrak{U}$ ; we must show that  $X \setminus A \in \mathfrak{U}$ . There exists  $U \in \mathfrak{U}$  such that  $U \cap A = \emptyset$ ; otherwise  $\mathfrak{U} \cup \{U\}$  has the finite intersection property and generates a filter which properly refines  $\mathfrak{U}$ , contrary to the maximality of  $\mathfrak{U}$ . Thus  $U \subseteq X \setminus A$  which implies that  $X \setminus A \in \mathfrak{U}$  as required. Thus  $\mathfrak{U}$  is an ultrafilter. □

It is instructive to consider an ultrafilter  $\mathfrak{U}$  on  $[0, 1]$  which extends the filter of Example 1.3. Such an ultrafilter gives rise to a measure  $\mu$  on  $[0, 1]$  defined by

$$\mu(A) = \begin{cases} 1, & \text{if } A \in \mathfrak{U}; \\ 0, & \text{if } X \setminus A \in \mathfrak{U}. \end{cases}$$

By definition this measure satisfies

- $\mu(\emptyset) = 0, \quad \mu([0, 1]) = 1;$
- $\mu$  is monotone, i.e.  $A \subseteq B$  implies  $\mu(A) \leq \mu(B);$
- $\mu$  is finitely additive, i.e.  $\mu(A \cup B) = \mu(A) + \mu(B)$  whenever  $A, B \subseteq [0, 1]$  are disjoint (although in general  $\mu$  is not countably additive, by the remarks following the proof of Theorem 2.5);
- $\mu(A) = 0$  whenever  $\lambda(A) = 0.$

Unlike Lebesgue measure  $\lambda(A)$  which is only defined for *certain* subsets  $A \subseteq [0, 1]$ , our measure  $\mu$  is defined for *all* subsets  $A \subseteq [0, 1]$ . In exchange for this desirable feature, we sacrifice something: the measure  $\mu$  does not agree with Borel measure on intervals  $[a, b] \subseteq [0, 1]$  (for which the Borel measure is simply the length  $b-a$ ).

**Theorem 2.2.** Let  $\mathfrak{U}$  be an ultrafilter on a set  $X$ . If  $\mathfrak{U}$  contains a finite set  $S$  then  $\mathfrak{U} = \mathfrak{F}_{\{s\}}$  for some  $s \in X$ .

**Corollary 2.3.** If  $X$  is a finite set then every ultrafilter on  $X$  is principal.

*Proof of Theorem 2.2.* Suppose  $\mathfrak{U}$  contains a finite set, and let  $S \in \mathfrak{U}$  with  $|S|$  minimal. Suppose  $|S| > 1$  and let  $s \in S$ . By minimality of  $|S|$  we must have  $\{s\} \notin \mathfrak{U}$ ; but then  $X \setminus \{s\} \in \mathfrak{U}$  and so  $S \setminus \{s\} = S \cap (X \setminus \{s\}) \in \mathfrak{U}$ , again contradicting the minimality of  $|S|$ . It follows that  $S = \{s\}$  and so  $\mathfrak{U} \supseteq \mathfrak{F}_{\{s\}}$ . Since  $\mathfrak{F}_{\{s\}}$  is an ultrafilter we must have  $\mathfrak{U} = \mathfrak{F}_{\{s\}}$ .  $\square$

**Theorem 2.4.** Let  $\mathfrak{U}$  be an ultrafilter on an infinite set  $X$ . Then  $\mathfrak{U}$  is nonprincipal iff  $\mathfrak{U}$  refines the finite complement filter of Example 1.2.

*Proof.* If  $\mathfrak{U}$  is principal then  $\mathfrak{U} = \mathfrak{F}_{\{s\}}$  for some  $s \in X$  by Theorem 2.2; but then  $X \setminus \{s\} \notin \mathfrak{U}$  where  $X \setminus \{s\}$  is in the finite complement filter. We may therefore suppose  $\mathfrak{U}$  is nonprincipal, and let  $S \subseteq X$  be nonempty and finite. By Theorem 2.2, we must have  $S \notin \mathfrak{U}$  and so  $X \setminus S \in \mathfrak{U}$ . Thus  $\mathfrak{U}$  refines the finite complement filter.  $\square$

**Theorem 2.5.** Let  $\mathfrak{U}$  be an ultrafilter on  $X$ , and let  $X = X_1 \cup X_2 \cup \cdots \cup X_n$  be a partition (so the  $X_i$ 's are mutually disjoint). Then there exists  $i \in \{1, 2, \dots, n\}$  such that  $X_i \in \mathfrak{U}$  and  $X_j \notin \mathfrak{U}$  for all  $j \neq i$ .

*Proof.* For  $n = 1$  there is nothing to prove. Assume the result holds for some  $n \geq 1$  and let  $X = X_1 \cup X_2 \cup \cdots \cup X_n \cup X_{n+1}$  be a partition. By the inductive hypothesis exactly one of

$$X_1, X_2, \dots, X_{n-1}, X_n \cup X_{n+1}$$

is in  $\mathfrak{U}$ . If  $X_n \cup X_{n+1} \in \mathfrak{U}$  but  $X_n \notin \mathfrak{U}$  then  $X_{n+1} = (X \setminus X_n) \cap (X_n \cup X_{n+1}) \in \mathfrak{U}$ .  $\square$

The analogue of Theorem 2.5 for infinite partitions does not hold; for example if  $\mathfrak{U}$  is a nonprincipal ultrafilter on an infinite set  $X$  then consider the partition of  $X$  into singleton subsets, none of which are in  $\mathfrak{U}$ .

### 3. Hyperreals

As usual  $\omega = \{0, 1, 2, 3, \dots\}$  and  $\mathbb{R}^\omega$  is the set of all functions  $\omega \rightarrow \mathbb{R}$ . We identify a function  $f : \omega \rightarrow \mathbb{R}$  with its infinite sequences of values:

$$f = (f(0), f(1), f(2), \dots).$$

Note that  $\mathbb{R}^\omega$  is a ring under pointwise addition and multiplication of functions; or equivalently, coordinatewise addition and multiplication of sequences:

$$\begin{aligned} (f(0), f(1), f(2), \dots) + (g(0), g(1), g(2), \dots) &= (f(0)+g(0), f(1)+g(1), f(2)+g(2), \dots); \\ (f(0), f(1), f(2), \dots)(g(0), g(1), g(2), \dots) &= (f(0)g(0), f(1)g(1), f(2)g(2), \dots). \end{aligned}$$

This is a commutative ring with zero element  $(0, 0, 0, \dots)$  and multiplicative identity  $(1, 1, 1, \dots)$ . The ring  $\mathbb{R}^\omega$  has zero divisors, e.g.

$$(0, 1, 1, 1, \dots)(1, 0, 0, 0, \dots) = (0, 0, 0, 0, \dots)$$

and so is clearly not a field. Let  $\mathfrak{U}$  be a nonprincipal ultrafilter on  $\omega$ , and say two functions  $f, g : \omega \rightarrow \mathbb{R}$  are *equivalent* (denoted  $f \sim g$ ) if there exists  $U \in \mathfrak{U}$  such that  $f(u)=g(u)$  for all  $u \in U$ . (In this case we say  $f$  and  $g$  agree ‘almost everywhere’.) Let  $\widehat{\mathbb{R}} = \mathbb{R}^\omega / \sim$ , the set of equivalence classes. We denote the equivalence class of  $f \in \mathbb{R}^\omega$  by

$$[f] = [(f(0), f(1), f(2), \dots)].$$

Two such sequences give the same element of  $\widehat{\mathbb{R}}$  iff they agree almost everywhere, i.e. on some  $U \in \mathfrak{U}$ . Addition and multiplication in  $\widehat{\mathbb{R}}$  are naturally defined by

$$[f] + [g] = [f + g]; \quad [f][g] = [fg].$$

We verify that these operations are well-defined: If  $f \sim f'$  and  $g \sim g'$ , then there exist  $U, V \in \mathfrak{U}$  such that  $f(u) = f'(u)$  for all  $u \in U$ , and  $g(v) = g'(v)$  for all  $v \in V$ . Then  $f(u)+g(u) = f'(u)+g'(u)$  for all  $u \in U \cap V \in \mathfrak{U}$  so  $f+g \sim f'+g'$  and similarly  $fg \sim f'g'$ .

In fact  $\widehat{\mathbb{R}}$  is a field. To verify this consider any  $[f] \neq [0]$ , so there exists  $U \in \mathfrak{U}$  such that  $f(u) \neq 0$  for all  $u \in U$ . Define

$$g(u) = \frac{1}{f(u)} \quad \text{for all } u \in U.$$

The definition of  $g(x)$  for  $x \in \omega \setminus U$  is irrelevant; we may take  $g(x) = 0$  for such values of  $x$  (this choice does not alter the equivalence class  $[g]$ ). Since  $f(u)g(u) = 1$  for all  $u \in U$ ,

we have  $[f][g] = [1]$  and so  $[g] = [f]^{-1}$ . To paraphrase this argument: if  $[f] \neq [0]$  then  $f$  is nonzero ‘almost everywhere’ so  $1/f(x)$  is defined ‘almost everywhere’, and this suffices to define an inverse for  $[f] \in \widehat{\mathbb{R}}$ . The reader with a background in algebra should observe that in fact  $\widehat{\mathbb{R}}$  is a quotient of the ring  $\mathbb{R}^\omega$  by the maximal ideal consisting of all functions  $f$  that vanish on some set in  $\mathfrak{U}$ . (If we had used instead a principal ultrafilter then this quotient ring would have been simply  $\mathbb{R}$  rather than  $\widehat{\mathbb{R}}$ .)

We regard  $\mathbb{R} \subset \widehat{\mathbb{R}}$  as embedded via the diagonal embedding  $x \mapsto [(x, x, x, \dots)]$ . Clearly  $\mathbb{R}$  is a subfield of  $\widehat{\mathbb{R}}$  in this way.

Next we check that  $\widehat{\mathbb{R}}$  is a totally ordered field. Given  $f, g : \omega \rightarrow \mathbb{R}$ , let  $X_1, X_2, X_3 \subseteq \omega$  be the set of all elements  $x \in \omega$  such that  $f(x) < g(x)$ ,  $f(x) = g(x)$  or  $f(x) > g(x)$  respectively. By Theorem 2.5, exactly one of  $X_1, X_2, X_3$  is in  $\mathfrak{U}$  and we define  $[f] < [g]$ ,  $[f] = [g]$  or  $[f] > [g]$  respectively. If  $[f] < [g]$  and  $[h] > [0]$ , then  $f < g$  almost everywhere and  $h > 0$  almost everywhere; thus  $fh < gh$  almost everywhere and so  $[f][h] = [fh] < [gh] = [g][h]$ .

An example of an infinitesimal in  $\widehat{\mathbb{R}}$  is  $\varepsilon = [(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)] \in \widehat{\mathbb{R}}$ . By definition we have  $\varepsilon > [0]$ . For every  $n \geq 1$  we have  $[(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)] < [(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots)]$  since the corresponding entries satisfy this inequality almost everywhere, i.e. on  $\{n, n+1, n+2, \dots\} \in \mathfrak{U}$ . (Recall from Theorem 2.4 that  $\mathfrak{U}$  refines the finite complement filter.) Thus  $\varepsilon$  is positive, but less than every positive standard real number.

An example of an infinite quantity in  $\widehat{\mathbb{R}}$  is  $\alpha = [(0, 1, 2, 3, \dots)]$ . For every  $n \geq 1$  we have  $[(0, 1, 2, 3, \dots)] > [(n, n, n, n, \dots)]$ ; thus  $\alpha$  exceeds every positive standard real.

The finer properties of  $\widehat{\mathbb{R}}$  are not explicitly answered by our construction. For example  $\beta = [(1, 0, 3, 2, 5, 4, 7, 6, \dots)] \in \widehat{\mathbb{R}}$ , like  $\alpha$ , exceeds every positive standard real; but which of  $\alpha$  and  $\beta$  is larger depends on the choice of ultrafilter used. In fact  $\alpha = \beta \pm 1$  and the choice of sign depends on which of the two sets  $A = \{0, 2, 4, 6, \dots\}$  or  $\omega \setminus A = \{1, 3, 5, 7, \dots\}$  belongs to  $\mathfrak{U}$ . This choice, in fact, determines whether  $[(0, 1, 0, 1, 0, 1, 0, 1, \dots)]$  equals  $[0]$  or  $[1]$ . We may arrange this choice in advance; for example by choosing  $\mathfrak{U}$  to be an ultrafilter which contains  $\mathfrak{F} \cup \{A\}$  we may guarantee that  $A \in \mathfrak{U}$ . But then what about  $\gamma = [(1, 1, 1, 4, 4, 4, 7, 7, 7, \dots)]$ ? To decide how the infinite value  $\gamma$  compares to  $\alpha$  and  $\beta$  depends on the choice of ultrafilter used. We cannot explicitly specify in advance what the answers to all such questions will be since this would require us to make uncountably many arbitrary choices.

Every element of  $\widehat{\mathbb{R}}$  is either infinite (i.e. larger than every standard real, loosely represented by the symbol  $\infty$ ; examples include  $\alpha$  and  $\beta$ ) or negative infinite (i.e. less than every standard real, loosely represented by the symbol  $-\infty$ ) or is ‘infinitely close’ to some real number (e.g.  $5 + \varepsilon$  is ‘infinitely close’ to 5). This is surprising since you might try to take an apparently random sequence with wildly oscillating terms; yet if  $(f(0), f(1), f(2), \dots)$  is any such sequence, consider  $A_f = \{a \in \mathbb{R} : [a] < [f]\}$ . Then  $s = \sup A_f \in [-\infty, \infty]$  and according to the value of this supremum, either  $[f]$  is infinite ( $\pm\infty$ ; note that  $s = \infty$  if  $A_f = \mathbb{R}$ ;  $s = -\infty$  if  $A_f = \emptyset$ ) or  $s \in \mathbb{R}$  with  $[f]$  ‘infinitely close’ to  $s$ .

Although the precise arithmetic of  $\widehat{\mathbb{R}}$  depends on the choice of ultrafilter used, one can ask whether any two choices of (nonprincipal) ultrafilter nevertheless give *isomorphic*



versions of the hyperreals. The answer is affirmative *if the Continuum Hypothesis is assumed*; recall that this is the assumption that there is no cardinal number strictly between  $\aleph_0$  and  $2^{\aleph_0}$ .

It is not hard to see that our field of hyperreals has the same cardinality as the reals. On the one hand  $|\widehat{\mathbb{R}}| \geq |\mathbb{R}| = 2^{\aleph_0}$  since  $\widehat{\mathbb{R}} \supset \mathbb{R}$ . On the other hand,

$$|\widehat{\mathbb{R}}| = |\mathbb{R}^\omega / \sim| \leq |\mathbb{R}^\omega| = |\mathbb{R}|^{|\omega|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$$

using basic properties of cardinal arithmetic. This gives equality:  $|\widehat{\mathbb{R}}| = 2^{\aleph_0}$ . An alternative construction of the ‘hyperreals’ uses the filter on  $\mathbb{R}^{[0,1]}$  given in Example 1.3. This leads to another field satisfying (N1)–(N3), larger than the model we have constructed. It is not possible to distinguish this new model from our  $\widehat{\mathbb{R}}$  by any set of first-order axioms; that is, if one writes down the set  $T$  of (first-order) axioms for  $\widehat{\mathbb{R}}$ , then there exist models of  $T$  having arbitrarily large cardinality.

Contained within the hyperreal number system  $\widehat{\mathbb{R}}$  are the hypernatural numbers  $\widehat{\mathbb{N}}$ , the hyperintegers  $\widehat{\mathbb{Z}}$ , and the hyperrational numbers  $\widehat{\mathbb{Q}}$ , these being the ultrapowers of  $\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{Q}$  respectively. Many properties of  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  are easily deduced from their nonstandard counterparts; for example, a quick and slick proof of the infinitude of primes in  $\mathbb{N}$  follows from considering  $\widehat{\mathbb{N}}$ . Natural and intuitive proofs of many standard results in ordinary real analysis follow from basic properties of  $\widehat{\mathbb{R}}$ . For example one may prove the Intermediate Value Theorem for  $\mathbb{R}$  as follows: Let  $f$  be a continuous real-valued function on the real interval  $[a, b]$ , such that  $f(a) < 0 < f(b)$ . Divide  $[a, b]$  into  $n$  subintervals of equal width, where  $n \in \widehat{\mathbb{N}}$ . Then  $f$  changes sign on one of these subintervals. If  $n$  is infinite, then we have a subinterval of infinitesimal width on which  $f$  changes sign; and from this we obtain a real root of  $f$ . Likewise a Riemann integral may be obtained from an actual sum, but with  $n$  terms where  $n \in \widehat{\mathbb{N}}$  is infinite. One can also form the hypercomplex numbers  $\widehat{\mathbb{C}} \supset \mathbb{C}$  as an algebraically closed extension containing infinitesimals.

## 4. More General Ultraproducts

Let  $(M_x)_{x \in X}$  be an indexed family of models of some theory  $T$  in first-order logic. Think of  $T$  as a set of axioms, and each  $M_x$  is an example of some system that satisfies these axioms. For example if  $T$  is the set of axioms for ordered fields, then for every  $x \in X$  we have a particular ordered field  $M_x$ . The product  $\prod_{x \in X} M_x$  consists of all functions  $f : X \rightarrow \bigcup_{x \in X} M_x$  such that  $f(x) \in M_x$  for all  $x \in X$ . Given an ultrafilter  $\mathfrak{U}$  on  $X$ , we define two elements  $f, g$  of the product space to be *equivalent* if  $f$  and  $g$  agree when restricted to some member  $U \in \mathfrak{U}$ . The corresponding **ultraproduct** of the  $M_x$ ’s, denoted  $M = (\prod_{x \in X} M_x) / \mathfrak{U}$ , consists of the equivalence classes of elements of  $\prod_{x \in X} M_x$ . This is again a model of the theory  $T$ . In particular if  $M_x$  is independent of  $x \in X$  then we call  $M$  an **ultrapower**. Our construction of  $\widehat{\mathbb{R}}$  above is an ultrapower construction.

A nontrivial example in which the factors in the product space are not constant, is found by taking a countable ultraproduct of all the prime-order finite fields  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$  as

$p$  ranges over all primes. This gives a field  $F$  with many interesting properties; for example  $F$  has characteristic zero, and has a unique (up to isomorphism) field extension of each degree  $n \geq 1$ .

## 5. Example: Ultraproducts of Graphs

As a further example, let us consider ultraproducts of graphs. The language of graphs requires a single binary relation symbol for adjacency. The axioms for graph theory require that this relation be symmetric (at least for ordinary graphs, where edges are not directed) and irreflexive (if we do not want loops). If  $\{\Gamma_\xi\}_{\xi \in X}$  is an indexed family of graphs, and  $\mathcal{U}$  is an ultrafilter on  $X$ , then the ultraproduct  $\Gamma = (\prod_{\xi \in X} \Gamma_\xi)/\mathcal{U}$  is also a graph. If  $\Sigma$  is a set of first-order sentences in the language of graphs, and  $\Gamma_\xi \models \Sigma$  for all  $\xi \in X$ , then  $\Gamma \models \Sigma$ . For example, if every  $\Gamma_\xi$  is triangle-free (i.e. having no 3-cycles), then  $\Gamma$  is also triangle-free. It is promising to look for new and interesting graphs this way, in particular infinite graphs which share many properties of known families of finite graphs.

An ultraproduct of graphs of degree  $k$  has degree  $k$ . Here we can replace ‘degree  $k$ ’ by ‘degree at most  $k$ ’ or by ‘degree  $\in K$ ’ for any finite set  $K$  of natural numbers. Or we can replace ‘degree’ by ‘diameter’ or ‘girth’ throughout. In each case the property in question is expressible in first order logic.

However, connectedness is not preserved by ultraproducts, as this is not a first-order property. For example, if  $\Gamma_n$  is a path of length  $n$  for each  $n \geq 1$ , then a (nonprincipal) ultraproduct  $\Gamma = (\prod_{n \geq 1} \Gamma_n)/\mathcal{U}$  has uncountably many infinite paths as its connected components, each path having 0 or 1 (but never 2) endpoints. Although the property of having diameter 3 (or diameter  $k$  for any other fixed  $k$ ) is expressible in first order logic, yet we cannot quantify over  $k$ , only over vertices.

A (nonprincipal) ultraproduct of bipartite graphs need not be bipartite, as the property of being bipartite is not expressible in the first order theory of graphs. (One can of course use instead the first order theory of bipartite graphs, in which a unary relation is added to the language in order to distinguish the two parts of the partition.) More generally, the property of having chromatic number  $k$  is not a first-order property of graphs, and so this property need not be preserved by ultraproducts. (The case of chromatic number 2 is equivalent to the property of bipartiteness.) For example if  $K_2$  is the graph on two vertices with one edge, then  $K_2$  has chromatic number 2 (i.e. is bipartite), whereas a nonprincipal ultrapower of  $K_2$  is a complete graph on an uncountable vertex set, so its chromatic number is uncountable.

## 6. A Proof of the Compactness Theorem

Following Cameron’s book, we obtained that the Compactness Theorem for first order logic as an easy consequence of the Soundness and Completeness Theorem. However, we did not work through the details of the proof of the Soundness and Completeness Theorem.

Here, for the first time in class this semester, we give a reasonably self-contained proof of the Compactness Theorem for first order logic.

Let  $\Sigma$  be a set of first order sentences in some language  $L$ . Let  $X$  be the collection of all finite subsets of  $\Sigma$ . We suppose that for each  $A \in X$ , there exists a model  $M_A \models A$ . We must show that  $M \models \Sigma$  for some  $L$ -structure  $M$ . (Without loss of generality,  $\Sigma$  is infinite; otherwise the desired conclusion follows immediately.) We will obtain  $M$  as an ultraproduct over  $X$ . However, we must be careful in the choice of ultrafilter  $\mathcal{U}$  on  $X$ ; it is not sufficient for  $\mathcal{U}$  to be nonprincipal. Also note that we require an ultrafilter on  $X$  rather than on  $\Sigma$  itself.

For each  $A \in X$ , consider the collection of all finite supersets of  $A$ :

$$A^+ = \{B \in X : B \supseteq A\} \subseteq X.$$

Note that for any  $A_1, A_2, \dots, A_n \in X$ , we have  $A_1 \cup A_2 \cup \dots \cup A_n \in X$  since this is a finite union of finite subsets of  $X$ , so

$$A_1^+ \cap A_2^+ \cap \dots \cap A_n^+ = (A_1 \cup A_2 \cup \dots \cup A_n)^+ \neq \emptyset.$$

Since the collection  $\{A^+ : A \in X\}$  of subsets of  $X$  satisfies the finite intersection property, it generates a filter  $\mathfrak{F}$  on  $X$ . Extend  $\mathfrak{F}$  to an ultrafilter  $\mathcal{U} \supseteq \mathfrak{F}$  on  $X$ , and consider the ultraproduct  $M = (\prod_{A \in X} M_A) / \mathcal{U}$ .

We must show that  $M \models \Sigma$ . Equivalently, for every finite subset  $B \subseteq \Sigma$ , we show that  $M \models B$ . Clearly  $M_A \models B$  for all  $A \in B^+$ , since in this case  $M_A \models A \supseteq B$ . Note that  $B^+ \in \mathcal{U}$  since  $B^+ \in \mathfrak{F}$ . We have  $M_A \models B$  for almost all  $A \in X$  (with respect to the ultrafilter  $\mathcal{U}$ ) so  $M \models B$  as required.