# Background to Cherlin's Paper

G. Eric Moorhouse, University of Wyoming

We begin with a quick review of the background from model theory needed to follow Cherlin's proof [2] that every generalized quadrangle with at most five points per line is finite. Further details on model theory can be found in [1], [3] (for general topics); for the more specialized notion of order indiscernibles, [3] is probably the best source.

## Prerequisites from Model Theory

We start with a language $\mathcal{L}$ containing special symbols for constants, relations and functions. Each relation or function symbol has a fixed 'arity' (the number of arguments) which is 2 for binary relations/functions, 1 for unary functions/relations, etc. By the language of $\mathcal{L}$, we mean statements of first-order logic over $\mathcal{L}$; this allows for taking conjunction, disjunction, negation, conditional statements, and quantifying over elements of some universal set (but not over *subsets* of the universal set, or *functions* defined on the universal set, etc.).

**Example 1: Ordered Fields.** The theory of ordered fields is expressed using the language $\mathcal{L}$ of ordered rings, as follows. Here we have two constant symbols '0' and '1'; two binary operation symbols '+' and '×'; one unary operation symbol '−'; and one binary relation symbol '<'. In this language we may write down the axioms for an ordered field, including the usual commutative, associative and distributive laws; also the axiom

$$(\forall x)(x \neq 0 \to (\exists y)(xy = 1))$$

to say that nonzero elements are invertible. Include also the required properties for '<', such as

$$(\forall x)(\forall y)(\forall z)(x < y \to x + c < y + c).$$

Let $\mathcal{A}$ be the set of all such axioms for an ordered field. We write $\mathbb{R} \vDash \mathcal{A}$ ('$\mathbb{R}$ models $\mathcal{A}$') to say that $\mathbb{R}$ satisfies all the statements in $\mathcal{A}$ under the usual interpretation of the symbols for constants, operations and relations. (Strictly speaking, a *model* $\mathcal{M}$ consists of an underlying set $M$, *and* rules for interpreting the various language symbols: interpreting constant symbols as particular elements of $M$, interpreting relation symbols as actual relations on $M$, and interpreting function symbols as actual functions on $M$.) Similarly, $\mathbb{Q} \vDash \mathcal{A}$. Note that the set $\mathcal{A}$, consisting of axioms for an ordered field, is finite.

One of the most useful results of model theory is

**Compactness Theorem.** *Let $\Sigma$ be a set of first-order statements over a language $\mathcal{L}$. If every finite subset $\Sigma_0 \subseteq \Sigma$ is satisfiable (i.e. has a model), then $\Sigma$ is satisfiable.*

For example, let $\mathcal{L}$ be as above, but with an additional constant symbol $\varepsilon$. For each $n \geqslant 1$, consider the statement

$$T_n : \qquad 0 < \underbrace{\varepsilon + \varepsilon + \cdots + \varepsilon}_{n \text{ times}} < 1.$$

Adjoining these statements to our axioms $\mathcal{A}$ for an ordered field gives the infinite set $\Sigma = \mathcal{A} \cup \{T_1, T_2, T_3, \ldots\}$. Every finite subset of $\Sigma$ is satisfiable, by realizing $\varepsilon$ as a sufficiently small positive real number. The Compactness Theorem guarantees the existence of an ordered field $F$ containing infinitesimals, as well as infinite elements.

Up to now, all variables appearing in statements were *bound*. However, we may consider also *formulas* containing *free* variables. A formula $\phi(x_1, x_2, \ldots, x_n)$ denotes a statement involving free variables $x_1, x_2, \ldots, x_n$, and all other variables are bound. For example, the formula

$$\phi(x_1, x_2) : \qquad (\forall u)(\exists v)(ux_1 = vx_2)$$

has two bound variables '$u$' and '$v$', and two free variables '$x_1$' and '$x_2$'. For any field $F$ and any $a, b \in F$, we have

if $b \neq 0$, then $F \vDash \phi(a, b)$;
if $b = 0$, then $F \nvDash \phi(a, b)$.

## Ramsey's Theorem

This theorem appears in two forms, finite and infinite. The infinite version is easier to state and easier to prove (using the pigeonhole principle); and the finite version is easily deduced from it.

**Ramsey's Theorem (Finite Version).** *Let $k, m, r \geqslant 1$. There exists $N = N(k, m, r)$ such that for every set $X$ with $|X| \geqslant N$ and every $k$-colouring of the $r$-subsets of $X$, there is an $m$-subset of $X$, all of whose $r$-subsets have the same colour.*

**Ramsey's Theorem (Infinite Version).** *Let $k, r \geqslant 1$, and let $X$ be an infinite set. For every $k$-colouring of the $r$-subsets of $X$, there is an infinite subset $A \subseteq X$, all of those $r$-subsets have the same colour.*

We obtain the finite version from the infinite version by contradiction, using the Compactness Theorem. For details, see [1], [3]. The difficulty with the finite case is the preponderance of quantifiers. This phenomenon also accounts for the role of model theory in Cherlin's proof: without model theory, the proof would be possible in principle, yet the proof would require so many additional quantifiers as to make the proof overly daunting to the reader, and much more elusive to the aspiring prover.

## Generalized Quadrangles

One possible approach to axiomatizing the theory of generalized quadrangles starts with a language $\mathcal{L}$ containing

- Two unary relations $P$ and $L$. Elements of our structure (i.e. points and lines of our GQ) will come from one universal set. The intended interpretation of the statement $P(x)$ is that $x$ is a point; similarly $L(x)$ is intended to mean that $x$ is a line.
- One binary relation $I$ representing incidence between points and lines.

(In the case of GQ's, we will not require any function symbols. An alternative approach starts with binary function symbols for 'meet' and 'join'; but we don't consider that approach here. And we don't need any constant symbols yet.) The language of GQ's is a first-order theory over this simple language, i.e. there exists a set $\mathcal{A}$ of statements in first-order logic which can be taken as the axioms for generalized quadrangles. We can take a set $\mathcal{A}$ of axioms including

(A0) $(\exists x)(\exists y)(P(x) \wedge L(y))$

(A1) $(\forall x)((P(x) \vee L(x)) \wedge \neg(P(x) \wedge L(x))$

(A2) $(\forall x)(\forall y)(I(x,y) \rightarrow (P(x) \wedge L(y)))$

(A3) $(\forall x)(\forall y)((P(x) \wedge L(y) \wedge \neg I(x,y)) \rightarrow (\exists u)(\exists v)(I(x,u) \wedge I(v,u) \wedge I(v,y)))$

(A4) $(\forall x)(\forall y)(\forall u)(\forall v)((I(x,u) \wedge I(x,v) \wedge I(y,u) \wedge I(y,v)) \rightarrow (x = y \ \vee \ u = v))$

Axiom (A0) says that there exists a point and a line. Axiom (A1) says that every object is either a point or a line, but never both. Axiom (A2) says that the incidence relation is satisfied only for point-line pairs. Axiom (A3) says that the incidence graph has diameter 3. Axiom (A4) says that the incidence graph has no 4-cycles. In order to guarantee thickness, we may require

(A5) $(\forall x)(P(x) \rightarrow (\exists y_1)(\exists y_2)(\exists y_3)(y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3 \wedge I(x,y_1) \wedge I(x,y_2) \wedge I(x,y_3)))$

(A6) $(\forall y)(L(y) \rightarrow (\exists x_1)(\exists x_2)(\exists x_3)(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3 \wedge I(x_1,y) \wedge I(x_2,y) \wedge I(x_3,y)))$

so that $\mathcal{G} \models \{(A0),(A1),\ldots,(A6)\}$ iff $\mathcal{G}$ is a (thick) generalized quadrangle. We shall require more: for some fixed $k \geqslant 3$, we want every line to have exactly $k$ points. Thus, in place of (or in addition to) axiom (A6), we require

(A6′) $(\forall y)(L(y) \rightarrow (\exists x_1)(\exists x_2)\cdots(\exists x_k)(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \cdots \wedge x_{k-1} \neq x_k \wedge I(x_1,y) \wedge I(x_2,y) \wedge \cdots \wedge I(x_k,y) \wedge (\forall x)(I(x,y) \rightarrow (x = x_1 \vee x = x_2 \vee \cdots \vee x = x_k))))$

where the missing symbols '$\cdots$' may be given explicitly in terms of $k$. If we fix $k \geqslant 3$ and take $\mathcal{A} = \{(A0), (A1), \ldots, (A6')\}$, then Cherlin's Theorem says that $\mathcal{A}$ has no infinite models: if we assume

(*) there exists an infinite model $\mathcal{G} \models \mathcal{A}$,

then we may obtain a contradiction. In order to understand Cherlin's argument, we need one more key idea, that of order indiscernibles, explained below; and the assumption that $\mathcal{G}$ is infinite will be needed in order to satisfy the main theorem on existence of order indiscernibles.

The assumption that there are infinitely many lines, or equivalently (given our other axioms) that every point lies on infinitely many lines, was not stated as an axiom; rather, it appeared as a 'meta-statement'. If we had wanted formal axioms stating that there are infinitely many lines, we could have used an infinite list of axioms of the form

(A7$_1$) $(\exists y_1)(L(y_1))$
(A7$_2$) $(\exists y_1)(\exists y_2)(L(y_1) \wedge L(y_2) \wedge y_1 \neq y_2)$
(A7$_3$) $(\exists y_1)(\exists y_2)(\exists y_3)(L(y_1) \wedge L(y_2) \wedge L(y_3) \wedge y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3)$
etc.

We avoided this approach since it is an unnecessary complication. (Note that the use of an infinite list of axioms here compensates for the fact that we can quantify only over elements of the universal set, not over its subsets.)

## Order Indiscernibles

Non-existence proofs often make use of symmetry in order to say that if a certain object exists, then without loss of generality, such-and-such happens. This symmetry is often supplied by a group of automorphisms. In place of automorphisms, we can use the weaker notion of order indiscernibles to reduce the number of cases under consideration: any argument that applies in the other cases, works just as well in the case we are considering. Let's explain...

Consider an arbitrary language $\mathcal{L}$ and theory $\mathcal{A}$, having a model $\mathcal{M} \models \mathcal{A}$ with underlying set $M$. We say that a subset $S \subseteq M$ is a set of *indiscernibles* if for any two $r$-tuples $s = (s_1, s_2, \ldots, s_r)$ and $s' = (s'_1, s'_2, \ldots, s'_r)$ of distinct elements of $S$, and for every first-order formula $\phi(x_1, x_2, \ldots, x_r)$ over $\mathcal{L}$, we have $\phi(s)$ iff $\phi(s')$.

**Example 2: Algebraically Independent Sets.** Let $\mathcal{L}$ be the language of rings with identity, and let $\mathcal{A}$ be the axioms for a field. Then $\mathbb{C} \models \mathcal{A}$. Any algebraically independent set $S$ is a set of indiscernibles. Consider the three formulas

$\phi_1(x_1, x_2, x_3)$: $(x_1x_2)x_3 = x_1(x_2x_3)$

$\phi_2(x_1, x_2)$: $\quad x_1^2 + x_1 x_2 + x_2^2 = 0$

$\phi_3(x_1, x_2)$: $\quad (\forall u)(\exists v)(ux_1 + vx_2 = 1)$

Then $\phi_1(s)$ is true for every triple $s$ from $S$. Because of algebraic independence, $\phi_2(s)$ is false and $\phi_3(s)$ is true for every pair $s$ from $S$.

The latter example arises because every permutation of $S$ extends to an automorphism of the field $\mathbb{C}$. However, indiscernibility does not necessarily arise from automorphisms of the model, as the following example shows.

**Example 3: A Union of Cliques.** Let $\mathcal{L}$ be the language of graphs, which includes one binary relation for adjacency; and let $\mathcal{A}$ be the axioms required for graphs asserting that the binary relation is symmetric and irreflexive). Consider the graph $\Gamma \vDash \mathcal{A}$ consisting of a disjoint union of cliques $\Gamma_j$ for $j$ ranging over some index set $J$. We may choose the cliques $\Gamma_j$ in such a way that $\Gamma_j$ has $\gamma_j$ vertices, where $\gamma_j$ are chosen to be distinct infinite cardinals. Let $S = \{s_j : j \in J\}$ where $s_j$ is a vertex chosen from $\Gamma_j$. Then $S$ is a set of indiscernibles in $\Gamma$. (In our first-order theory of graphs, we can express the fact that a graph has infinitely many vertices, but we cannot distinguish one infinite cardinality from another.) Clearly, however, no nontrivial automorphism of $S$ extends to an automorphism of the graph $\Gamma$.

**Example 4a: The Rational Order.** Consider the language of ordered sets with the binary relation symbol '$<$', and let $\mathcal{A}$ be the set of axioms for a total order (i.e. linear order). Then $\mathbb{Q} \vDash \mathcal{A}$. Here we regard $\mathbb{Q}$ as an ordered set, without regard for the ring operations. In this case, there does not exist a set $S$ of indiscernibles with $|S| \geqslant 2$. To see this, let $\phi(x_1, x_2)$ be the formula $x_1 < x_2$. For every pair $(a_1, a_2)$ of distinct elements of $\mathbb{Q}$, one of the two statements $\phi(a_1, a_2)$,$\phi(a_2, a_1)$ is true and the other is false. The obstacle to finding sets of indiscernibles is evidently the order relation.

From the latter example, we see that the order relation is an obstacle to obtaining indiscernibles. This motivates the following definition:

Let $\mathcal{L}$, $\mathcal{A}$ and $\mathcal{M}$ be as above, and let $J$ be a totally ordered set. Consider an indexed set $S = \{s_j : j \in J\}$ of elements of $\mathcal{M}$. We say $S$ is a set of *order indiscernibles* (of *order type* $J$) if for every pair of $r$-tuples $(j_1, j_2, \ldots, j_r)$ and $(j'_1, j'_2, \ldots, j'_r)$ with $j_1 < j_2 < \cdots < j_r$ and $j'_1 < j'_2 < \cdots < j'_r$ in $J$, and every first-order formula $\phi(x_1, x_2, \ldots, x_r)$ over $\mathcal{L}$, we have $\phi(s_{j_1}, s_{j_2}, \ldots, s_{j_r})$ iff $\phi(s_{j'_1}, s_{j'_2}, \ldots, s_{j'_r})$.

The following is found in [3, p.179]. We paraphrase the proof here, since this will provide a context in which we can more easily describe some needed extensions of this theorem.

**Theorem A (Existence of Order Indiscernibles).** *If $\mathcal{A}$ has infinite models, then for every order type $J$, $\mathcal{A}$ has a model containing a set $S$ of indiscernibles of order type $J$.*

*Proof.* Adjoin new constant symbols $c_j$ for $j \in J$ to obtain the new language $\mathcal{L}^* = \mathcal{L} \cup \{c_j : j \in J\}$. Form the new set of statements $\Sigma$ consisting of

(i) all statements in $\mathcal{A}$;
(ii) for all $j \neq j'$ in $J$, the statement $c_j \neq c_{j'}$; and
(iii) for all formulas $\phi(x_1, x_2, \ldots, x_r)$ over $\mathcal{L}^*$ and all pairs of increasing sequences $j_1 < j_2 < \cdots < j_r$ and $j'_1 < j'_2 < \cdots < j'_r$ in $J$, the statement $\phi(c_{j_1}, c_{j_2}, \ldots, c_{j_r}) \rightarrow \phi(c_{j'_1}, c_{j'_2}, \ldots, c_{j'_r})$.

Any model of $\Sigma$ will be a model of $\mathcal{A}$ over the original language, with $\{c_j : j \in J\}$ realized as a set of indiscernibles of order type $J$. So it suffices to show that $\Sigma$ has a model. Suppose a finite subset $\Sigma_0 \subset \Sigma$ is given; by the Compactness Theorem, it suffices to show that $\Sigma_0$ has a model. By hypothesis, there exists a model $\mathcal{M} \vDash \mathcal{A}$ with an infinite set of elements $M$. Arbitrarily choose a total order $<$ on $M$; this will allow us to identify every finite subset $A \subset M$ with an $r$-tuple of elements listed in increasing order.

Consider a finite set of formulas of type (iii) (those appearing in $\Sigma_0$), which we may write as $\phi_i(x_1, x_2, \ldots, x_r)$ for $i = 1, 2, \ldots, m$ (after pooling together all the free variables appearing in $\phi_1, \phi_2, \ldots, \phi_m$). We will colour all the $r$-subsets of $M$ using $2^m$ colours, as follows: To each $r$-subset $A = \{a_1, a_2, \ldots, a_r\} \subset M$ with $a_1 < a_2 < \cdots < a_r$, we assign the colour $\nu(A) := (\nu_1(A), \nu_2(A), \ldots, \nu_m(A)) \in \{T, F\}^m$ where $\nu_i(A) = T$ or $F$ according as $\phi_i(a_1, a_2, \ldots, a_r)$ holds, or does not hold, in $\mathcal{M}$. By (the infinite form of) Ramsey's Theorem, there exists an infinite subset $S \subseteq M$ such that $\nu(A) = \eta \in \{T, F\}^m$ is constant for all $r$-subsets $A \subset S$.

Now let $\{c_j : j \in J_0\}$ be the set of all $c_j$'s appearing in $\Sigma_0$; here $J_0 \subset J$ is a finite subset. Take any subset $S_0 \subset S$ having the finite cardinality $|J_0|$. There is a unique order-preserving bijection $\theta : \{c_j : j \in J_0\} \rightarrow S_0$; then we obtain from $\mathcal{M}$ a model for $\Sigma_0$, by realizing the symbol $c_j$ as $\theta(c_j) \in S_0$ for all $j \in J_0$. The fact that

$$\mathcal{M} \vDash \phi_i(c_{j_1}, c_{j_2}, \ldots, c_{j_r}) \leftrightarrow \phi_i(c_{j'_1}, c_{j'_2}, \ldots, c_{j'_r})$$

for all $i = 1, 2, \ldots, m$ and all pairs of $r$-tuples $j_1 < j_2 < \cdots < j_r$ and $j'_1 < j'_2 < \cdots < j'_r$ in $J_0$, follows from our choice of $S$. □

Several extensions of this result are possible. For example (see [3, p.180]), it is possible to exert greater control over the order relation on the set $S$, beyond simply prescribing its order type. In Example 5, we will show how to effectively use another extension of Theorem A.

**Example 4b: The Rational Order.** As before, let $\mathcal{L}$ be the language of ordered sets, let $\mathcal{A}$ be the axioms for a total order, and consider the model $\mathbb{Q} \models \mathcal{A}$. Then the entire set $\mathbb{Q}$ is a set of order indiscernibles in $\mathbb{Q}$. (Take $J = \mathbb{Q}$ with the usual order, and $s_j = j$ for each $j \in \mathbb{Q}$, giving $S = \mathbb{Q}$.) This follows from the fact that if $\rho = (j_1, j_2, \ldots, j_r)$ and $\rho' = (j'_1, j'_2, \ldots, j'_r)$ are two $r$-tuples of rationals with $j_1 < j_2 < \cdots < j_r$ and $j'_1 < j'_2 < \cdots < j'_r$, then there is an order-preserving bijection $\mathbb{Q} \to \mathbb{Q}$ mapping $\rho$ to $\rho'$.

The latter example arises because of the presence of a large group of automorphisms of the ordered set of rationals. However, order indiscernibles do not necessarily arise from actual automorphisms of the model; in other words, order-preserving permutations of the set $S$ of order indiscernibles, do not necessarily extend to automorphisms of the model $\mathcal{M}$. (See Example 3 and the remarks preceding it.) This can be remedied—see the remarks following Example 5.

**Example 5: Order Indiscernible Lines in GQ's.** Fix $k \geqslant 3$. Let $\mathcal{L}$ be the language of GQ's, as above; and let $\mathcal{A}$ be the axioms for semifinite GQ's with $k$ points per line, and infinitely many lines. Assuming that $\mathcal{A}$ has an infinite model, then $\mathcal{A}$ has a model containing a set $S$ of order indiscernibles of order type $\mathbb{Q}$. (Here $J = \mathbb{Q}$ with the usual order relation.) By order indiscernibility, and using the fact that each line has only finitely many points, we see that one of the following three conditions must hold:

(a) $S$ is a set of points of $\mathcal{M}$, no two collinear; or
(b) $S$ consists of a set of lines of $\mathcal{M}$, all passing through a common point $O$; or
(c) $S$ consists of a *partial spread*, i.e. a set of mutually disjoint lines of $\mathcal{M}$.

In order to apply Cherlin's argument, we must have case (c). Probably the easiest way to ensure this is as follows. In the proof of Theorem A (taking $\mathcal{A}$ to be our axioms for GQ's with line size $k$ as indicated above), add to our set $\Sigma$ the following statements:

(iv) for each $j \in J$, the statement $L(c_j)$; and
(v) for all $i < j$ in $J$, the statement $\neg(\exists x)(I(x, c_i) \wedge I(x, c_j))$.

We require the fact that every infinite GQ with line size $k$ must have an infinite set of mutually disjoint lines, i.e. an infinite partial spread; but this is easy to see. The rest of the proof of Theorem A goes through (details are left as an exercise) and we conclude that our set $S$ of indiscernibles satisfies condition (c) above.

If $\mathcal{G}$ satisfies (*) and $S$ is a partial spread with the order indiscernible property (as in Cherlin's proof), then consider the subquadrangle $\mathcal{H} \subseteq \mathcal{G}$ generated by $S$ (i.e. the lines of $S$, and the points covered by these lines) under the operations of 'meet' and 'join'. Then every order-preserving permutation of $S$ extends to an automorphism of $\mathcal{H}$. (The original quadrangle $\mathcal{G}$ need not have any automorphisms, however.) This follows from

[3, p.180]. We omit the details here since we do not require automorphisms; all that is needed for Cherlin's proof is the weaker condition of order indiscernibility. In terms of our original chosen formal language $\mathcal{L}$, which did not include the binary operations 'meet' and 'join', these two operations are examples of *Skolem functions.* In order to realize $\mathcal{H}$ we first *Skolemize* the language $\mathcal{L}$ by adjoining all Skolem functions; we then obtain the *Skolem hull* of $S$ (which in our case is $\mathcal{H}$) as the closure of $S$ under the Skolem functions. Cherlin's remarks in [2] show that he takes the viewpoint that we are dealing with the quadrangle $\mathcal{H}$ rather than $\mathcal{G}$, and so the role of indiscernibility is subsumed by the group of automorphisms of $\mathcal{H}$ which preserve $S$.

A *Skolem function* is used to provide witnesses to existentially quantified formulas. In particular if $\mathcal{M} \models T$ where $T$ is a set of sentences containing

$$(\forall x_1)(\forall x_2)\cdots(\forall x_n)(\exists y)(\phi(x_1, x_2, \ldots, x_n, y)),$$

then we can add an $n$-ary function symbol $f$ to our language, and there exists a model $\mathcal{M}^*$ satisfying

$$\mathcal{M}^* \models T \cup \{(\forall x_1)(\forall x_2)\cdots(\forall x_n)(\phi(x_1, x_2, \ldots, x_n, f(x_1, x_2, \ldots, x_n)))\}$$

i.e. $f$ provides a witness to the verity of the existentially quantified formula $\phi$. We have mentioned 'meet' and 'join' in a GQ as an example.

A more subtle example is the following. Every GQ $\mathcal{G}$ satisfies

$$\mathcal{G} \models (\forall y)(L(y) \to (\exists x)(I(x, y))),$$

i.e. every line has at least one point. (This conforms to the above form since it is equivalent to $\mathcal{G} \models (\forall y)(\exists x)(L(y) \to I(x, y)).$) So without loss of generality, we can assume that our language contains a function $f_1$ supplying a particular point on any given line:

$$\mathcal{G} \models (\forall y)(L(y) \to I(f_1(y), y)).$$

But there exists another point on the line $y$ different from $f_1(y)$, which must have the form $f_2(y)$ for yet another Skolem function $f_2$. Continuing in this manner, we obtain $f_1(y)$, $f_2(y)$, ..., $f_k(y)$ as the $k$ points of $y$. Now given any indiscernible set of lines $S$, it is easy to see that for each $i \in \{1, 2, \ldots, k\}$, we consider the point sets $U_i = \{f_i(s) : s \in S\}$ for $i = 1, 2, \ldots, k$. Then one of the following must hold:

- All lines in $S$ are concurrent. In this case, one of the sets $U_i$ is a singleton, and each of the others is an indiscernible set of points forming a cap. Or
- $S$ is a partial spread. In this case each of the sets $U_i$ is an indiscernible set of points forming a cap.

Similar arguments apply in the dual setting, allowing us to construct indiscernible sets of lines from indiscernible sets of points. This provides another solution to the hurdle encountered in Example 5 (trying to get from an arbitrary indiscernible set of objects, to an indiscernible set of lines forming a partial spread).

**Example 6: Including a Fixed Line $\ell_0$.** Cherlin's argument also makes use of a fixed line $\ell_0$, and he obtains a set $S = \{a_j : j \in \mathbb{Q}\}$ of lines not containing $\ell_0$, such that $S \cup \{\ell_0\}$ is a partial spread, and $S$ consists of indiscernibles *with respect to* $\ell_0$. This means that for every first-order formula $\phi(x_0, x_1, \ldots, x_r)$ over $\mathcal{L}$, and every pair of $r$-tuples $j_1 < j_2 < \cdots < j_r$ and $j'_1 < j'_2 < \cdots < j'_r$ in $\mathbb{Q}$, we have

$$\mathcal{M} \vDash \phi(\ell_0, a_{j_1}, a_{j_2}, \ldots, a_{j_r}) \leftrightarrow \phi(\ell_0, a_{j'_1}, a_{j'_2}, \ldots, a_{j'_r}).$$

The easiest way to accomplish this is probably to obtain a set $S = \{a_j : j \in \mathbb{Q}\}$ of indiscernible lines as before (in Example 5), and then to choose $\ell_0 = a_0$, and re-index the lines $S' = \{a_j : j \in \mathbb{Q},\ j > 0\}$ using a new index set $\mathbb{Q}$, which is order-isomorphic to the positive rationals.

An alternative approach, which may be helpful in other situations, is to introduce to the language a new constant symbol $c$, and to add to $\Sigma$ additional statements of the form $c \neq c_j$ and $\phi(c, c_{j_1}, \ldots, c_{j_r}) \leftrightarrow \phi(c, c_{j'_1}, \ldots, c_{j'_r})$ as necessary. We omit the details.

The last example concludes our background preparation for Cherlin's paper. We conclude with another application of order indiscernibles, which may be of use in attempts to extend Cherlin's argument.

**Example 7: Spreads, Caps and Ovoids in Semifinite GQ's.** It is not hard to see that every infinite GQ with finite line size has spreads. Cherlin's order indiscernible set consists of an infinite partial spread...what if instead this were a spread? (I don't see how to show the existence of a spread of order indiscernible lines just from (*), so this would be a new hypothesis.)

It is easy to see the existence of infinite caps (a *cap* being just a set of points, no two collinear); hence by Theorem A (suitably extended as in Example 5), we see that (*) implies the existence of an infinite cap with the order indiscernible property. Does this help in trying to extend Cherlin's argument? I don't see any way to show the existence of ovoids, just using (*). But again, we could ask what happens if we add the hypothesis of an order indiscernible ovoid—once again, does this get us any closer to extending Cherlin's argument?

## References


[1] P. J. Cameron, *Sets, Logic and Categories*, Springer-Verlag, London, 1998.

[2] G. Cherlin, 'Locally finite generalized quadrangles with at most five points per line', *Disc. Math.* **291** (2005), 73–79.

[3] D. Marker, *Model Theory: An Introduction*, Springer-Verlag, New York, 2010.