Knots

(Handout April 25, 2012)

Adjoints

Let A be an $n \times n$ matrix. The (i, j)-*minor of* A is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the i-th row and j-th column of A. The (i, j)-*cofactor of* A equals $(-1)^{i+j}$ times the (i, j)-minor of A. The *matrix of cofactors of* A is the $n \times n$ matrix adj(A) whose (i, j)-entry is the (i, j)-cofactor of A. The *adjoint of* A, denoted adj(A), is the transpose of the matrix of cofactors of A. For example, you may check that the adjoint of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 5 \\ -2 & 3 & 1 \end{bmatrix}$$

is given by

$$\mathrm{adj}(A) = \begin{bmatrix} -16 & -3 & 15 \\ -12 & 1 & -5 \\ 4 & -9 & -7 \end{bmatrix}.$$

Adjoints are useful because of the fact that

$$A \cdot \mathrm{adj}(A) = (\det A)I = \mathrm{adj}(A) \cdot A.$$

From this it follows that if $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj}(A).$$

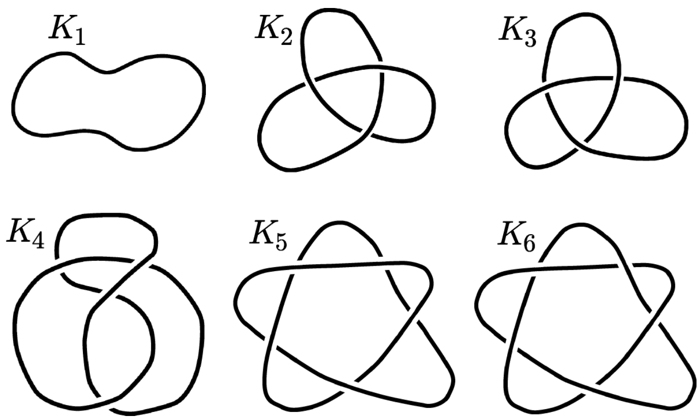
In the example above, we obtain $\det(A) = -52$ and so

$$A^{-1} = -\frac{1}{52}\begin{bmatrix} -16 & -3 & 15 \\ -12 & 1 & -5 \\ 4 & -9 & -7 \end{bmatrix}.$$

Knots

Informally, a **knot** is a closed loop of string. You can make a knot from any piece of string by connecting its two ends together, thereby making it ‘closed’. (This is easily done with a cord as demonstrated in class.) The simplest knot is the **unknot**, obtained by connecting two ends of the string together without first tangling the string. More complicated knots are formed by tangling the string before connecting its ends together. Two knots are considered the same if one can be transformed to the other by manipulating the string but without breaking apart or disconnecting either knot.

We cannot include a real knot in these notes, since a knot is 3-dimensional. However, we can show a 2-dimensional representation of a knot, called a **knot diagram**. Here are some examples of knot diagrams:



For example, K_1 represents the unknot. Diagrams K_2 and K_3 represent the *left-handed* and *right-handed trefoil knots.* We say that K_2 and K_3 are *mirror images,* for obvious reasons. At each crossing point of a knot diagram, we have been careful to show (by a break in the sketch for the lower portion of string) which part of the string passes underneath the other.

Note that two different knot diagrams can possibly represent the same knot. For example, knot diagrams K_2 and K_4 represent the same knot. Two knot diagrams K, K' are *equivalent* if they represent the same knot; in this case we write $K \sim K'$. Observe that '$\sim$' is an equivalence relation on the set of all possible knot diagrams. Among the diagrams above, we have $K_4 \sim K_2$, and $K_5 \sim K_1$. These equivalences are easily found using your piece of string. You may try to convince yourself (using your piece of string) that K_1, K_2, K_3 and K_6 are inequivalent (in which case the diagrams shown represent four equivalence classes of knots). However, this is not possible by naïve manipulation of knots. While simple manipulation of string shows that (for example) $K_5 \sim K_1$, it is not possible to show $K_2 \not\sim K_1$ in this way. After all, how do we really know we aren't just too dumb to find the right manipulation? Couldn't someone else come along and find a clever way of manipulating K_2 into K_1 or K_6? This cannot be done, as J.W. Alexander showed in 1928.


James Waddell Alexander II
1888–1971

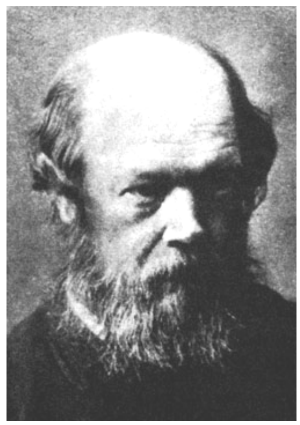
To each knot diagram K, Alexander associated a polynomial $A_K(x) \in \mathbb{Z}[x]$ which is an *invariant* of K in the sense $A_{K'}(x) = A_K(x)$ whenever $K' \sim K$. We will compute the Alexander polynomials of the knots above to obtain

$$\begin{aligned} A_{K_1}(x) &= 1, & A_{K_4}(x) &= 1 - x + x^2, \\ A_{K_2}(x) &= 1 - x + x^2, & A_{K_5}(x) &= 1, \\ A_{K_3}(x) &= 1 - x + x^2, & A_{K_6}(x) &= 1 - x + x^2 - x^3 + x^4. \end{aligned}$$

From this it will follow that K_6 is not equivalent to any of the other knot diagrams above. Unfortunately, Alexander polynomials are unable to distinguish between mirror images. It is true that $K_2 \not\sim K_3$ (this was proved by M. Dehn in 1914), but to show this requires something more than Alexander polynomials.

Before describing how to compute Alexander polynomials, we mention that knots (and their associated polynomials) have immense significance in theoretical physics. As long ago

William Thomson (Lord Kelvin)
1824–1907



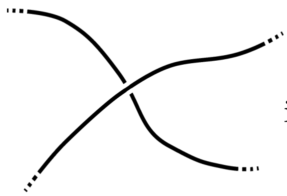
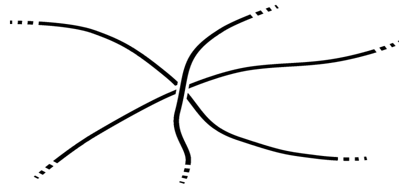
Peter Guthrie Tait
1831–1901

Vaughan Frederick Randall Jones
1952–

as the 19th century, this was suspected, when William Thomson (better known as Lord Kelvin) conjectured that chemical properties of the elements could be explained by viewing atoms as knots in ether. This motivated P.G. Tait to begin a systematic classification of knots in 1877. Although Kelvin's view did not stand the test of time, modern theoretical physics (especially topological quantum field theory) is strongly related to the theory of knots and their associated polynomials, including Alexander and Jones polynomials. In 1990, V. Jones received the Fields Medal (the highest possible award in mathematics) for his recent work which led to relationships between knot theory, quantum statistical mechanics, quantum field theory, and the prediction of DNA configurations in certain biological interactions.

Computing Alexander Polynomials

We must assume that at most two portions of the string cross at any crossing point of a knot diagram; thus

 is allowed;  is *not* allowed.

An *arc* of a knot diagram is a segment with two endpoints, which is drawn continuously without lifting the pencil. For example, K_2 and K_3 each have three arcs (as well as three crossing points). K_4 has four arcs (and four crossing points). Both K_5 and K_6 have five arcs (and five crossing points). A careful reading of the definition shows that K_1 has no arcs (and no crossing points).

Proposition 1. Any knot diagram has the same number of arcs as crossing points.

Proof. Let K be any knot diagram, and let N be the number of endpoints of arcs in K. Clearly $N = 2 \times$ (number of arcs) since by definition, each arc has two endpoints. Also, exactly 2 endpoints abut at each crossing point (according to the restriction at the beginning of this section). Therefore, $N = 2 \times$ (number of crossing points). Equating these two expressions for N and canceling 2's gives the required result. □

The following is an algorithm for computing the Alexander polynomial of any given knot diagram K.

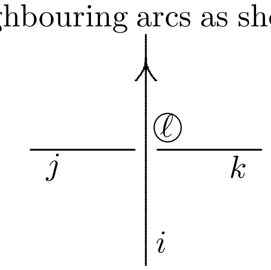
Step 1. Draw the knot diagram K, large and clear.

Step 2. Label its arcs 1, 2, …, n, and its crossing points ①, ②, …, ⓝ. (This is possible by Proposition 1.)

Step 3. Orient the diagram. This means to give a direction to K by placing arrows along the sketch of the string, as though we were indicating the direction of flow of liquid or electricity along the loop. (It doesn't matter which of the two possible orientations, or directions, we choose, as long as we stick to one orientation throughout.)

Step 4. Write down an $n \times n$ matrix of zeroes. Label its rows 1, 2, …, n, and its columns ①, ②, …, ⓝ.

Step 5. For every crossing point ⓛ, with direction and neighbouring arcs as shown, modify the matrix of Step 4 as follows:



In column ⓛ of the matrix,

add $1-x$ to the entry in row i (i.e. the (i,ℓ)-entry);

add -1 to the entry in row j (i.e. the (j,ℓ)-entry);

add x to the entry in row k (i.e. the (k,ℓ)-entry).

Call the resulting $n \times n$ matrix A.

Step 6. Compute any minor or cofactor of A. This gives a polynomial in x. (Note: We define the determinant of a 0×0 matrix to be 1.)

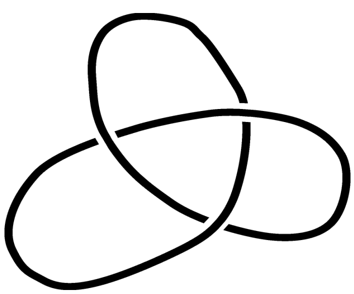
Step 7. If the polynomial in Step 6 has no constant term, say x^r is the lowest degree term appearing, then divide this polynomial by x^r. The resulting polynomial has a nonzero constant term.

Step 8. If the polynomial resulting from Step 7 has a negative constant term, multiply the polynomial by -1; otherwise leave it unchanged.

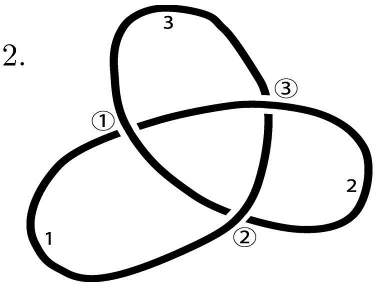
The polynomial resulting from Step 8 is the *Alexander polynomial of A,* and is denoted $A_K(x)$. It is clear by construction that $A_K(x)$ is a polynomial in x having integer coefficients.

As an example, we compute the Alexander polynomial of the left-handed trefoil knot K_2, following the above eight steps:

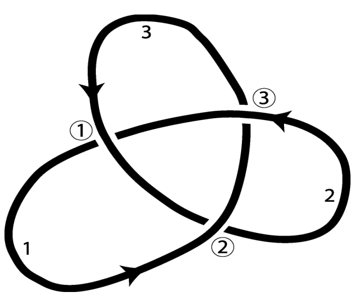
Step 1.



Step 2.



Step 3.



Step 4.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} \text{①} & \text{②} & \text{③} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Step 5.

$$A = \begin{array}{c} \begin{array}{ccc} \text{①} & \text{②} & \text{③} \end{array} \\ \begin{bmatrix} x & 1-x & -1 \\ -1 & x & 1-x \\ 1-x & -1 & x \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Step 6. The $(1,2)$-minor of A is

$$\begin{vmatrix} -1 & 1-x \\ 1-x & x \end{vmatrix} = -1 + x - x^2.$$

Step 7. $-1 + x - x^2$

Step 8. $A_{K_2}(x) = 1 - x + x^2.$

When the number n of crossings is large, great care is required to correctly compute the $(n-1)\times(n-1)$ minor in Step 6. It is useful to have a number of safeguards for checking whether the computations are correct. Here are some means of checking computations:

(i) A should be singular. If you are using `MAPLE`, check that $\det(A) = 0$.

(ii) The sequence of coefficients of $A_K(x)$ should be *palindromic,* i.e. read the same from left-to-right as right-to-left. (Examples of *palindromes* in English are "Madam in Eden, I'm Adam" and "Able was I ere I saw Elba".) Observe that the Alexander polynomials given for K_1 through K_6 each have this property.

(iii) The Alexander polynomial $A_K(x)$ is independent of the choice of minor used in Step 6. If working by hand, try a different minor. If working with `MAPLE`, asking for $\text{adj}(A)$ gives all cofactors of A (which are plus or minus the minors of A) and it should be easy to inspect $\text{adj}(A)$ (possibly after factoring entries) to see that all entries give the same final polynomial after repeating Steps 7 and 8. For instance, in the example above we have

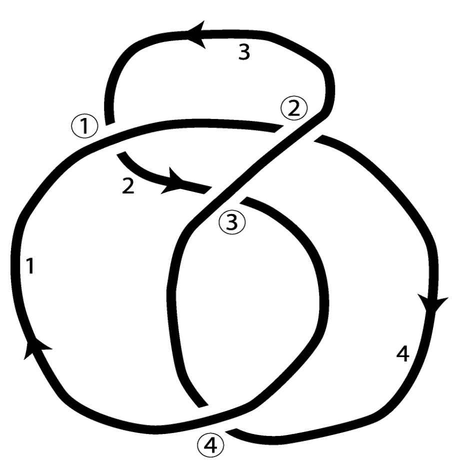
$$\text{adj}(A) = \begin{bmatrix} 1-x+x^2 & 1-x+x^2 & 1-x+x^2 \\ 1-x+x^2 & 1-x+x^2 & 1-x+x^2 \\ 1-x+x^2 & 1-x+x^2 & 1-x+x^2 \end{bmatrix}.$$

By inspection, all entries are the same, which provides a check that $1-x+x^2$ is the correct Alexander polynomial.

We will use the following theorem without proof.

Theorem 2. If K and K' are equivalent knot diagrams, then K and K' have the same Alexander polynomial. Therefore the Alexander polynomial depends only on the knot, not on the particular choice of diagram representing the knot.

For example, we compute the Alexander polynomial of K_4 :



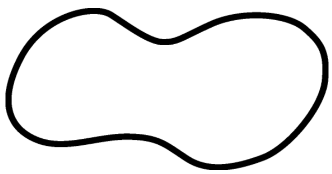
$$A = \begin{bmatrix} 1-x & -1 & x & 1-x \\ x & 0 & -1 & 0 \\ -1 & 1-x & 1-x & x \\ 0 & x & 0 & -1 \end{bmatrix},$$

$$\operatorname{adj}(A) = \begin{bmatrix} -1+x-x^2 & -1+x-x^2 & -1+x-x^2 & -1+x-x^2 \\ -1+x-x^2 & -1+x-x^2 & -1+x-x^2 & -1+x-x^2 \\ -x+x^2-x^3 & -x+x^2-x^3 & -x+x^2-x^3 & -x+x^2-x^3 \\ -x+x^2-x^3 & -x+x^2-x^3 & -x+x^2-x^3 & -x+x^2-x^3 \end{bmatrix},$$

and every entry of $\operatorname{adj}(A)$ gives the same polynomial $A_{K_4}(x) = 1 - x + x^2$.

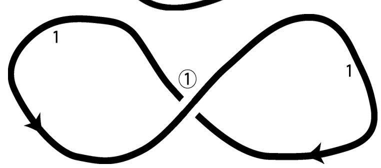
As further illustrations, consider the following diagrams equivalent to the unknot, all of which have Alexander polynomial 1:

K_1 :



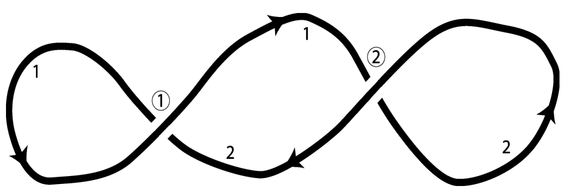
$\Rightarrow\ A = [\ \ ]$ (no entries) $\Rightarrow A_{K_1}(x) = 1$

K :



$\Rightarrow\ A = \big[\,1\,\big] \qquad \Rightarrow A_K(x) = 1$

K' :

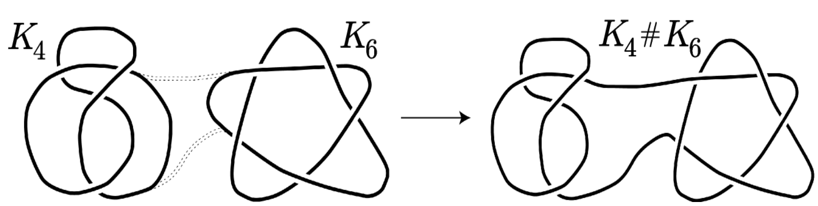


$\Rightarrow\ A = \left[\begin{smallmatrix} -x & x \\ x & -x \end{smallmatrix}\right], \quad \operatorname{adj}(A) = \left[\begin{smallmatrix} -x & -x \\ -x & -x \end{smallmatrix}\right]$

$\Rightarrow A_{K'}(x) = 1$

Connected Sums of Knots

Let's suppose you have two pieces of string such as you were given in class, and that you have formed knots K and K' with them. Disconnect both knots without disturbing the tangled portions of either knot, and join together the two free ends of K and K'. This forms a single knot, called the *connected sum* of K and K', denoted $K\#K'$. Here we show $K_4\#K_6$:

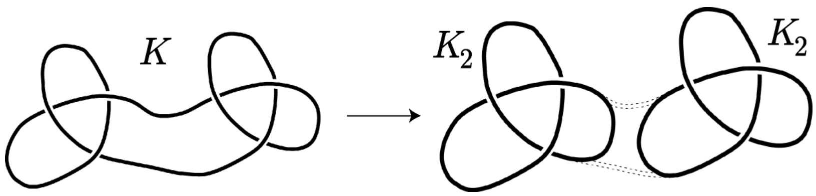


If K_1 is the unknot as before, then obviously $K\#K_1$ and $K_1\#K$ are both equivalent to K. Therefore the unknot is a 'two-sided identity' for the operation of connected sum (just like 0 is a two-sided identity for addition of integers, and 1 is a two-sided identity for multiplication of integers).

A knot K is called *prime* if anytime K is equivalent to $K'\#K''$, then at least one of the two factors K' or K'' is an unknot. (This is just like the definition of prime numbers in the positive integers!) The following analogue of the Fundamental Theorem of Arithmetic holds for knots (stated here without proof).

Theorem 3. Any knot K can be decomposed as a connected sum of prime knots. This decomposition is unique except for a possible reordering of the factors.

For example, the following knot K clearly factors as $K_2\#K_2$, where K_2 is the left-handed trefoil knot:



One computes in this case that

$$A_K(x) = A_{K_2\#K_2}(x) = (1-x+x^2)^2.$$

This is an illustration of the following amazing fact.

Theorem 4. If K and K' are any two knots, then $A_{K\#K'}(x) = A_K(x)A_{K'}(x)$.

Knot catalogues only list prime knots, since all other knots can be easily formed from these; and since their polynomials are readily computed using the polynomials of their prime factors.

Finally we remark that our use of polynomials in the context of knots, does not involve their use as functions; rather knot polynomials are used as polynomials in their own right!