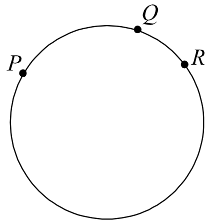
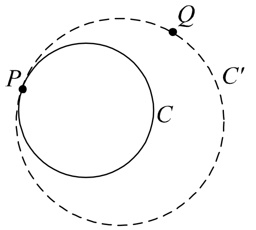
Inversive Plane Geometry

An *inversive plane* is a geometry with three undefined notions: *points*, *circles*, and an *incidence* relation between points and circles, satisfying the following three axioms:

(I.1) Through any three distinct points there is exactly one circle.



(I.2) If P and Q are points, and C is a circle passing through P but not Q, then there is a unique circle C' passing through Q such that $C \cap C' = \{P\}$.



(I.3) There exist four points which do not lie on a common circle.

A model of these axioms is provided by the points and circles lying on a sphere S in Euclidean 3-space. Another (obtained by stereographically projecting S onto a plane) is the extended Euclidean plane $E = \mathbb{R}^2 \cup \{\infty\}$ consisting of the Euclidean plane $\mathbb{R}^2$ together with one new point ∞ called *the point at infinity*. (This is *different* from the real projective plane in which there are *many* points at infinity.) The circles of E are of two types: the ordinary circles of $\mathbb{R}^2$, and sets of the form $\ell \cup \{\infty\}$ where ℓ is a line of $\mathbb{R}^2$. Because the second model is obtained from the first by stereographic projection, the two models are isomorphic. Other models exist (in particular finite models) but we will be primarily concerned with the model E described above, called the *real inversive plane*. In this case we may reasonably measure distances and angles.

Straightedge and Compass Constructions

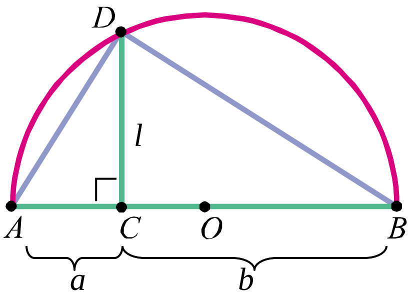
It is often instructive to provide, along with the relevant definitions, straightedge-and-compass constructions; and we shall often do so when this is feasible. Recall that the following procedures can be implemented using straightedge and compass:

1. Given a point P and a line ℓ, construct the line through P perpendicular to ℓ.
2. Find the midpoint of a line segment AB.
3. Bisect an angle ABC.

Using these basic constructions we can perform others, for example:

Lemma 1. Given line segments of lengths *a* and *b*, one may construct a line segment of length $\sqrt{ab}$. Thus given a rectangle, we may construct a square with the same area.

Proof. Construct a line segment AB containing a point C such that $AC = a$ and $BC = b$. Construct the midpoint O of AB. Construct a semicircle centered at O with radius $OA = OB$. Construct a perpendicular l to AB at C. Let D be the point of intersection of l with the semicircle. We will show that CD has the required length $\sqrt{ab}$. Observe that triangles ACD and DCB are similar since corresponding angles are equal. Therefore

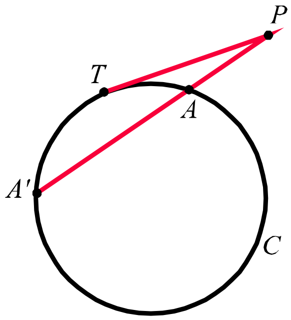


$$\frac{AC}{CD} = \frac{DC}{CB},$$

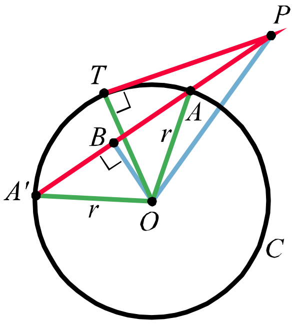
i.e. $CD^2 = AC \cdot CB = ab$ as required. $\square$

We will show that given a circle C and a point P outside C, one may construct a tangent from P to C. This construction relies on the following result.

Lemma 2. Let C be a circle and let P be a point outside C. Consider a tangent PT to C, and a secant through P meeting C at A and A'. Then $PA \cdot PA' = PT^2$.



Proof. Let O be the center of C, and let r be the radius. Drop a perpendicular OB from O to the secant, as shown. Then



$$\begin{aligned}
PT^2 &= OP^2 - OT^2 && \text{by Pythagoras' Theorem for triangle } PTO\\
&= OP^2 - OA^2 && \text{since } OA = OT = r\\
&= OP^2 - (OB^2 + AB^2) && \text{by Pythagoras' Theorem for triangle } ABO\\
&= (OP^2 - OB^2) - AB^2 &&\\
&= PB^2 - AB^2 && \text{by Pythagoras' Theorem for triangle } PBO\\
&= (PB + AB)\,(PB - AB) &&\\
&= PA' \cdot PA && \text{since } BA' = BA.
\end{aligned}$$

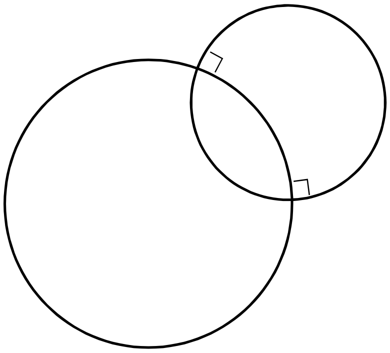
$\square$

Lemma 3. Given a circle C and a point P outside C, one may construct tangents from P to C.

Proof. Construct any secant to C through P, and let A and A' be the points of intersection of this secant with C. By Lemma 1 one may construct a line segment of length $\sqrt{PA' \cdot PA}$, which is the length of the required tangent. Set the radius of the compass to this length and draw an arc centered at P to intersect C at two points Q and R. Then PQ and PR are the required tangents. $\square$

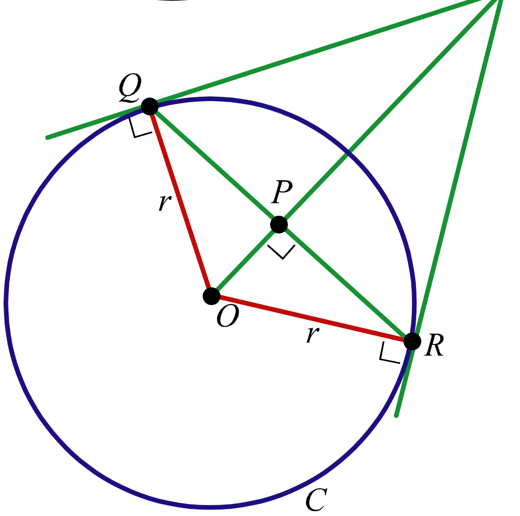
The Real Inversive Plane

Two circles in *E* are *orthogonal* (i.e. perpendicular) if they intersect at right angles. Note that in this case the circles meet twice, and if the angle at one point of intersection is 90°, the angle at the other point of intersection must also be 90°.



Two orthogonal circles

Given a circle *C* with center *O*, and a point *P*, we define the *inverse* *P′* of *P* in *C* as follows. The inverse of every point of *C* is itself ($P' = P$). The inverse of *O* is ∞, and the inverse of ∞ is *O*. If *P* is inside *C* (but different from *O*) then extend the line *OP* beyond the circle *C* and erect a perpendicular to this line at *P*. This perpendicular meets *C* at points *Q* and *R*, say. The tangents to *C* at *Q* and at *R* meet at *P′*. (Recall that the tangents are constructed as lines perpendicular to the radii *OQ* and *OR*.) Conversely the image of *P′* is *P*. In order to construct *P* given *P′*, we first join the line *OP′*. Construct the tangents from *P′* to *C* (see Lemma 3 for this construction). The line *QR* intersects *OP′* at the required point *P*.



The latter construction yields an algebraic formula for inversion: Note that the triangles *OPQ* and *OQP′* are similar, since they share a common angle at *O*, and the corresponding angles at *P* and *Q* (respectively) are right angles. Therefore corresponding sides of the two triangles are in the same proportion, so that

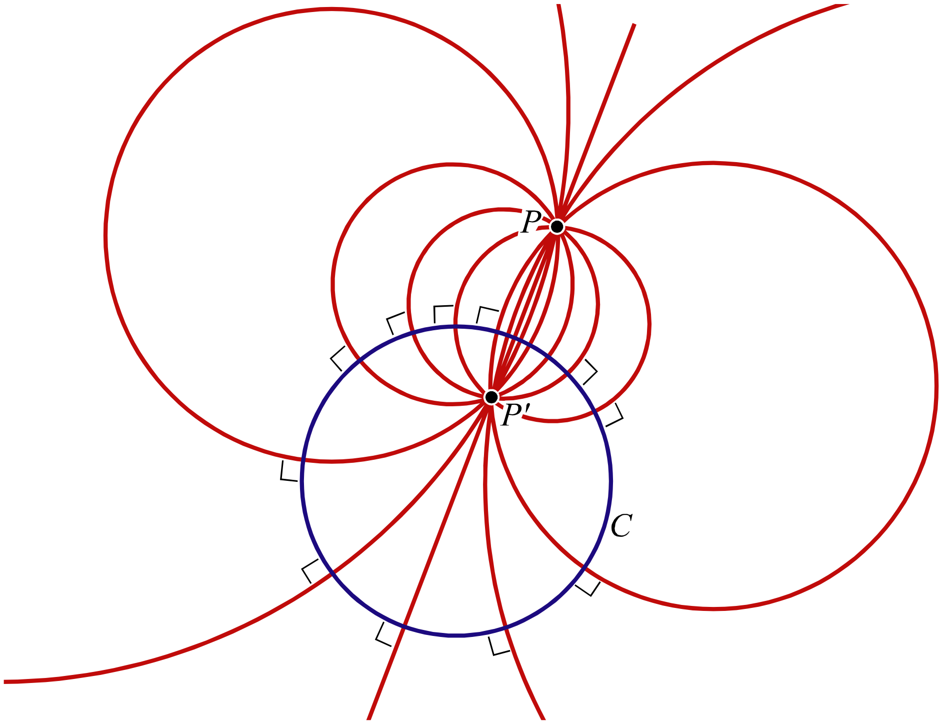
$$\frac{OP}{OQ} = \frac{OQ}{OP'} \ .$$

Since $OQ = r$ is just the radius of *C*, we obtain

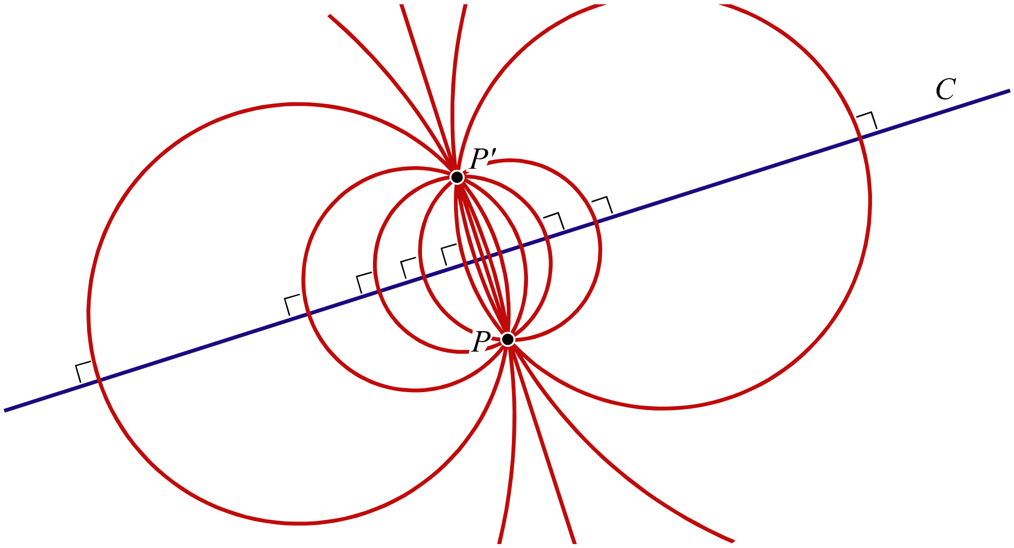
$$OP' = \frac{r^2}{OP} \, .$$

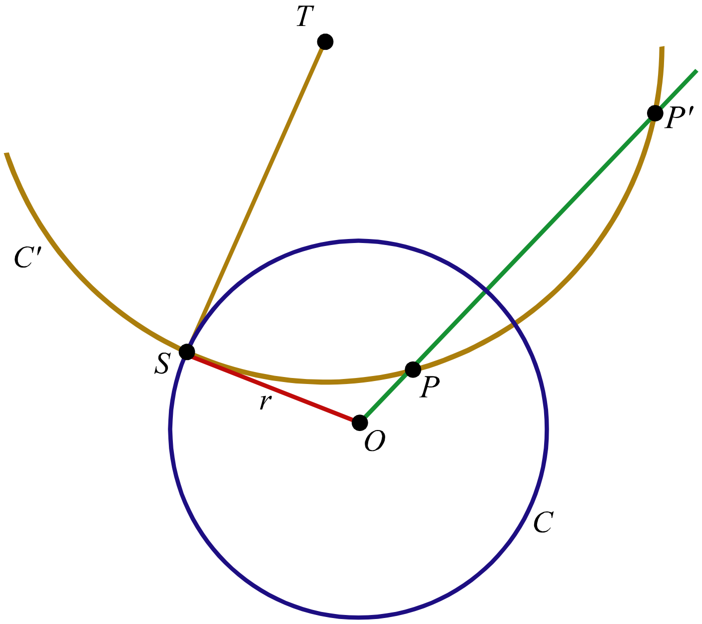
Given that *P′* lies on the line *OP*, the position of *P′* is uniquely determined by its distance from *O* as given by this formula. We now prove a remarkable fact about pairs of inverse points:

Theorem 1. Let *C* be a circle, and let *P* and *P′* be an inverse pair of points with respect to *C*. Then every circle through *P* and *P′* is orthogonal to *C*.



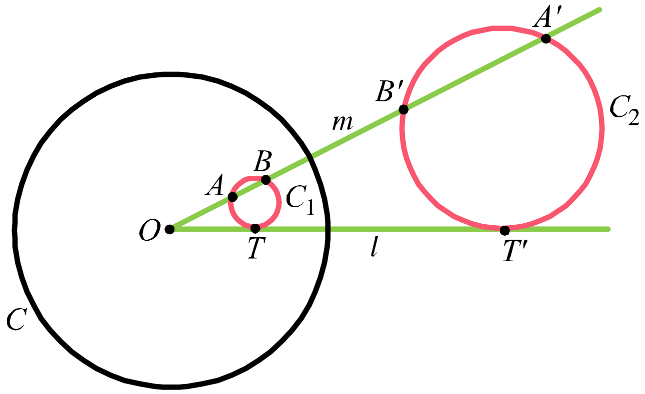
In the special case that *C* has infinite radius (i.e. *C* is a Euclidean line) then inversion is simply reflection in this line, and the points *P* and *P′* are mirror images in *C*.





Proof of Theorem 1. Consider any circle C' through P and P', and let T be the center of C'. Let S be a point of intersection of C' with T. In order to show that the circles C and C' are orthogonal, we only need to show that OS is tangent to C'. Since the points P and P' are inverse in C, we have $OP \cdot OP' = OS^2$. Therefore $OS = \sqrt{OP \cdot OP'}$ which, by Lemma 2, is exactly the length of the tangent from O to C'. Therefore OS is tangent to C' as required. □

Theorem 2. Inversion takes circles to circles.



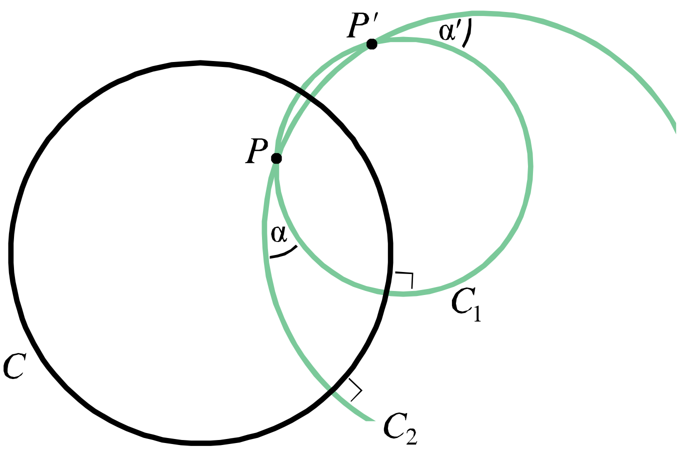
Proof. Let C be a circle with center O. Let A and A' be points inverted by C, and let T and T' be another pair of inverse points with respect to C. Let l be the line passing through O, T and T'. Let m be the line passing through O, A and A'. Let C_1 be the unique circle through A tangent to l at T, and let C_2 be the unique circle through A' tangent to l at T' as shown. Let B be the second point of intersection of m with C_1, and let B' be the second point of intersection of m with C_2. We must show that B' is the inverse of B with respect to C, i.e. $OB \cdot OB' = r^2$ where r is the radius of C.

By Lemma 2 we have $OT^2 = OA \cdot OB$ and $(OT')^2 = OA' \cdot OB'$. Since T and T' are inverse in C we have $OT \cdot OT' = r^2$, and similarly $OA \cdot OA' = r^2$. Therefore

$$r^4 = (OT)^2 \cdot (OT')^2 = OA \cdot OB \cdot OA' \cdot OB' = r^2\, OB \cdot OB'$$

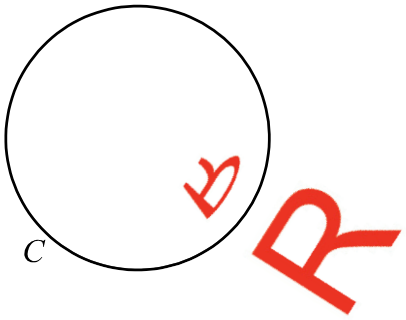
which yields $r^2 = OB \cdot OB'$ as required. □

Theorem 3. Inversion preserves angles.



Proof. Consider a pair of points P and P' inverted by a circle C, and let C_1 and C_2 be two circles through P and P'. Let α and α' be the angles between C_1 and C_2, at P and at P' respectively, as shown. (This really means the angles between the tangent lines to the circles C_1 and C_2, at P and P' respectively.) By Theorem 2, the inverse of C_1 in C is a circle through P and P', meeting C at the same points as C_1 does. But these points uniquely determine the circle C_1, so the inverse of C_1 with respect to C is C_1. Similarly the inverse of C_2 with respect to C is C_2. Therefore inversion takes angle α to angle α'. However these two angles must be the same size by symmetry, since they are the angles between circles C_1 and C_2 at their two points of intersection. Thus $\alpha' = \alpha$ as required. □

Note that inversion, like reflection, reverses orientation of plane figures. The accompanying figure shows a letter 'R' and its inverse image in the circle C; note that in addition to distances being distorted, the orientation of the 'R' has been reversed.



Interpreting Inversion in the Hyperbolic Plane

Let C and C' be a pair of orthogonal circles. Recall that the points of the plane interior to C represent points of the hyperbolic plane; and the arc of C' interior to C represents a line of the hyperbolic plane. Inversion in C' represents a reflection in the hyperbolic plane.