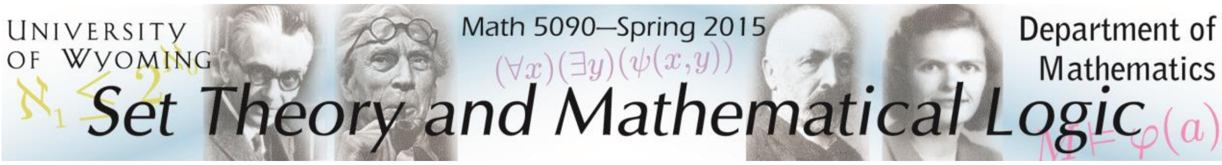

Countably vs. Uncountably Infinite

“Anti-Breugel” by Anatoly T. Fomenko (1976)

Created after the famous engraving “Alchemists” by Pieter Breugel. Rows of cups filled with molten metal receding to the horizon are used to convey the concept of infinity.

Contemplating infinity tends to perplex us, rather like the poor soul in the center of Fomenko’s work appears to be tormented. Certainly it can lead us to conclusions that appear paradoxical. But are such questions merely ‘mind games’ of pure abstraction, or do they have any practical relevance?

I have chosen to introduce some of the difficulties associated with the concept of infinity, by reference to the apparently practical question of determining the probabilities of real-life events.

Scenario I: A Pair of Dice

Consider rolling a pair of dice. The total of the values appearing on top is 2 (‘snake eyes’ 1+1=2) with probability $\frac{1}{36}$; or a value of 3 (arising as 1+2=3 or 2+1=3) with probability $\frac{2}{36}$; etc. The maximum total appearing is 12 (that’s 6+6=12) which occurs with probability $\frac{1}{36}$. The most likely value appearing is 7 (occurring in any of the six ways 1+6 = 2+5 = 3+4 = 4+3 = 5+2 = 6+1 = 7) which occurs with probability $\frac{6}{36}$. (It is simpler here to keep the common denominator of 36, rather than reducing the fractions). Notice that the sum of the probabilities of all these outcomes is

$$\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36}+\frac{5}{36}+\frac{6}{36}+\frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=1.$$

The concept of infinity does not arise in this situation. My purpose in describing this scenario is only to emphasize that the sum of the probabilities of all possible outcomes should equal 1.

Scenario II: Coin Flipping

You play a game in which a coin is flipped until the first head appears. How many flips are necessary? With probability $\frac{1}{2}$, a head appears on the first flip. With probability $\frac{1}{4}$, the first head appears on the second flip. With probability $\frac{1}{8}$, the first head appears on the third flip. And so on. There are infinitely many possible outcomes, and once again the corresponding probabilities add up to 1:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1.$$

The terms in this infinite sum correspond to the probabilities that the sequence of heads (H) and tails (T) starts out as H or TH or TTH or TTTH or TTTTH or TTTTTH or TTTTTTH or … To verify that the infinite sum actually does equal 1, we may denote the unknown value of the sum by x, thus:

$$x=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots.$$

But then multiplying both sides by 2 gives

$$2x=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots$$

and then subtracting the previous expression for x we see that all terms cancel except the term '1', leaving

$$2x-x=1$$

so that $x=1$. You should recognize by now that this is the well-known method for obtaining the sum of an infinite geometric series.

My point in describing this scenario is to emphasize that the principle that the sum of probabilities of all possible outcomes should equal 1, regardless of whether there are only finitely many possible outcomes, or infinitely many possible outcomes.

You might ask: haven't we omitted a term from the infinite sum above, namely the probability that a head *never* arises? This would mean that a coin flipped infinitely many times always gives tails (TTTTTTTTTTTT… forever). However, the probability of such an outcome is actually zero.

Scenario III: Tomorrow's Temperature

Tomorrow at 12:00 noon I will measure the temperature at the base of the flagpole in Prexy's Pasture. What is the probability that the temperature is exactly 37°F? Evidently this probability is zero. What is the probability that the temperature is exactly 38.2209843671°F? Again zero. How about exactly $10\pi = 31.4159265358979\ldots$°F? Again the probability is zero. It should be evident that for every specific value T that you select in advance, the probability that the temperature will be exactly T, is zero. But now the sum of these probabilities is zero (*not* 1). How do we resolve this paradox?

The difference between Scenario III and the previous scenarios, is that in Scenario III there are uncountably many possible outcomes. One cannot meaningfully interpret the sum of uncountably many real numbers.

One can meaningfully describe the probability that tomorrow's temperature lies in a specified interval, such as between 0°F and 10°F; but this requires integration rather than summation.