University of Wyoming — Math 4520—Fall 2014 — Algebra III — Fields — Department of Mathematics

Infinite-Dimensional Vector Spaces

The ring $\mathbb{R}[t]$ of all polynomials in t with real coefficients, is a vector space over the field $\mathbb{R}$. (Here we ignore the multiplication of polynomials, and only consider vector space operations: addition of polynomials, and multiplication by real scalars.) A natural basis for this vector space is $\mathfrak{B} = \{1, t, t^2, t^3, \ldots\}$. This means that every polynomial is uniquely expressible as a linear combination of monomials $t^j \in \mathfrak{B}$. For example, $(1+t)^4 \in \mathbb{R}[t]$ can be expressed as a linear combination of $1, t, t^2, t^3, t^4$:

$$(1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4.$$

When we say that this linear combination is unique, we are disregarding silliness like

$$(1+t)^4 = 4t + 6t^2 + 1 + 2t^3 + t^4 + 0t^5 + 2t^3;$$

this linear combination is essentially the same as the previous one, after simply permuting terms, adding terms with zero coefficients, or splitting up terms involving the same basis vector.

More generally, let V be a vector space over a field F. A *basis* for V is a subset $\mathfrak{B} \subset V$ such that every vector $v \in V$ can be uniquely expressed as a *linear combination*

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k$$

for some $k \geqslant 0$; $a_1, a_2 \ldots, a_k \in F$; and $v_1, v_2, \ldots, v_k \in \mathfrak{B}$. 'Uniqueness' is understood in the sense above: unique up to permuting terms, after collecting terms and disregarding terms with zero coefficient. Even in this sense, we may express the zero vector as a linear combination of $k = 0$ terms (since by definition, an empty sum equals zero). Note that by definition, a linear combination involves only finitely many terms; so convergence is not an issue. (It is possible to study topological vector spaces, but here we only consider vector spaces, an algebraic notion in which topological concerns of closure, convergence, etc. are an unnecessary complication.)

As further examples, the rings $\mathbb{R}(t)$, $\mathbb{R}[[t]]$ and $\mathbb{R}((t))$ are also vector spaces over $\mathbb{R}$ (see earlier handouts for definitions of these rings); but here it is not so clear whether in fact a basis is available in each of these cases.

Whenever $E \supseteq F$ is a field extension, we have learned to regard E as a vector space over F. For example, $\mathbb{C}$ is a vector space over $\mathbb{R}$, with basis $\{1, i\}$. Similarly, $\mathbb{R}$ is a vector space over $\mathbb{Q}$; but in this case it is not clear how to obtain a basis.

In the finite-dimensional case, finding a basis is straight-forward. Let V be a vector space over a field F. If $V = \{0\}$ then the empty set $\{\ \}$ is a basis for V. Otherwise, let $0 \neq v_1 \in V$. If $V = \langle v_1 \rangle$ then $\{v_1\}$ is a basis for V and we are done. Otherwise there exists a vector $v_2 \in V$ with $v_2 \notin \langle v_1 \rangle$. Now if $\langle v_1, v_2 \rangle = V$, then $\{v_1, v_2\}$ is a basis for V. If not, then there exists a vector $v_2 \in V$ with $v_3 \notin \langle v_1, v_2 \rangle$. Continue in this way. If V is finite-dimensional, the process will eventually terminate, giving a basis for V. But if V is infinite-dimensional, the process goes on indefinitely; and even after an infinite number of steps, there is no guarantee that we end up with a basis for V.

For example, consider the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. It is possible, by the process described in the previous paragraph, to obtain the set

$$\{1,\ \sqrt{2},\ \sqrt{3},\ \sqrt{5},\ \sqrt{6},\ \sqrt{7},\ \sqrt{10},\ \sqrt{11},\ \ldots\}.$$

This set is linearly independent, but it is not a basis for $\mathbb{R}$ over $\mathbb{Q}$: for example, π is not a linear combination of the numbers $\sqrt{m}$ with rational coefficients. Adding π to our set, the new set $\{\pi, 1, \sqrt{2}, \sqrt{3}, \ldots\}$ is still linearly independent; but it still fails to span all of $\mathbb{R}$. In fact, *there is no basis* of the form

$$\{v_1,\ v_2,\ v_3,\ \ldots\}$$

for $\mathbb{R}$ over $\mathbb{Q}$. The problem is that the sequence $v_1, v_2, v_3, \ldots$ is countably infinite; but the dimension of $\mathbb{R}$ over $\mathbb{Q}$ is uncountably infinite. This fact is not quite obvious, so let us explain:

Suppose V has a countably infinite basis $\mathfrak{B} = \{v_1, v_2, v_3, \ldots\}$ over $\mathbb{Q}$ (an example is $\mathbb{Q}[t]$ with basis $\{1, t, t^2, t^3, \ldots\}$) and consider the chain of subspaces

$$V_0 < V_1 < V_2 < V_3 < \cdots$$

where

$$\begin{aligned} V_0 &= \{0\}; \\ V_1 &= \langle v_1 \rangle; \\ V_2 &= \langle v_1, v_2 \rangle; \\ V_3 &= \langle v_1, v_2, v_3 \rangle, \end{aligned}$$

etc. so that $\dim V_n = n$ for all $n \geqslant 0$. Note that the vector space V_n is isomorphic to $\mathbb{Q}^n$, so it is countably infinite whenever $n \geqslant 1$. Since $\mathfrak{B}$ spans V, every $v \in V$ is a linear combination involving only finitely many basis vectors; so $v \in V_n$ for some $n \geqslant 0$. This says that

$$V = \bigcup_{n=0}^{\infty} V_n \;=\; V_0 \cup V_1 \cup V_2 \cup V_3 \cup \cdots.$$

Since V is a countable union of countable sets, V is countable. However, $\mathbb{R}$ is uncountable; so there is no countably infinite basis for $\mathbb{R}$ over $\mathbb{Q}$: the dimension of $\mathbb{R}$ over $\mathbb{Q}$ must be uncountable. How do we even begin to imagine constructing a basis for $\mathbb{R}$ over $\mathbb{Q}$?

Using Zorn's Lemma (as described in a previous handout), we may prove that *every vector space has a basis.* This argument will be described in class.

The latter result gives a solution to the following

Problem. Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

- $f(1) = 1$;
- $f(v + w) = f(v) + f(w)$ for all $v, w \in \mathbb{R}$; and
- $f(\sqrt{2}) = \pi$.