My Favourite Problem

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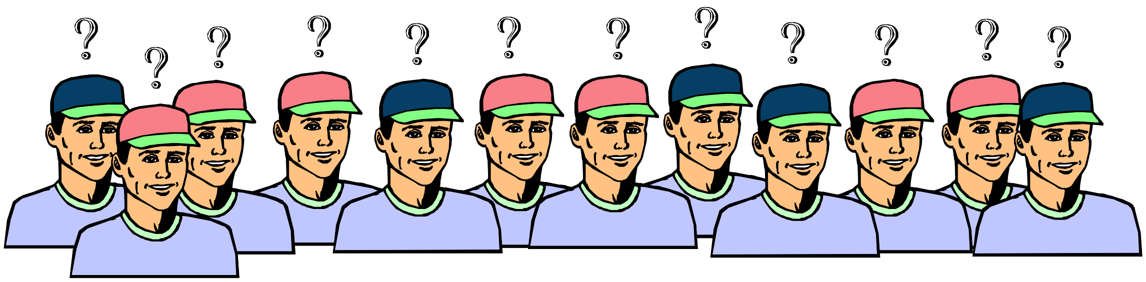
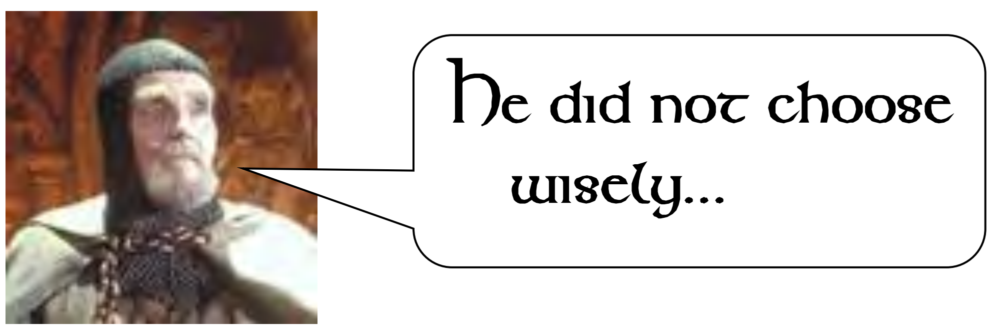
Infinitely many participants are gathered for a game in which I place on every participant's head a cap which is either red or blue.



Everyone can see the caps on the other participants' heads but cannot see their own cap. They must guess the colour of their own cap. If most (all but a finite number) of participants guess correctly, I suitably reward all participants. Otherwise (if infinitely many participants guess wrongly) I punish all participants by death.



Participants may plan a strategy before the contest starts but they are not permitted to communicate after I place the caps on their heads.

What strategy guarantees that the participants succeed?

Comments

1. I did not indicate that the number of participants is countably infinite. Assume this if you like (for the sake of concreteness) but the solution does not require this.
2. I did not reveal my strategy in assigning colours to the hats. For example, am I choosing colours randomly? You might as well assume

this: it is the worst case from the participants' point of view, because if I instead follow any deterministic plan for choosing colours, any such pattern could be perceived by the participants, which can only improve their chances of success.

Solution

Each assignment of hat colours is a function from the infinite set X of participants, to $\{R, B\}$ (abbreviating red=R, blue=B). Consider two such colour assignments

$$f, g : X \to \{R, B\}.$$

We say that f and g are *equivalent* (denoted $f \sim g$) if there are only finitely many $x \in X$ for which $f(x) \neq g(x)$. Clearly this is an equivalence relation on $\{R, B\}^X$, the set of all functions from X to $\{R, B\}$. For each equivalence class $[f]_\sim = \{g : g \sim f\}$, the participants (before hats are placed on heads) 'simply' choose a representative $\hat{f} \in [f]_\sim$ from that equivalence class, using the Axiom of Choice. After seeing all hats but his own, a typical participant sees the value of $f(y)$ for every $y \neq x$ where f is the actual assignment of hat colours. Although the value of $f(x)$ is unknown to him, he knows f to within equivalence. He recalls the representative $\hat{f} \in [f]_\sim$, and so guesses $\hat{f}(x)$ as the colour of his hat. By definition, only finitely many guesses are incorrect.

Further comments

1. According to this scheme, any single participant may be expected to guess incorrectly with probability ½ . You might suppose that given any n participants, the chances of all of them guessing correctly is 2^{-n}. This conclusion is erroneous, as it is based on the assumption of statistical independence of the guesses, which does not hold.
2. Assuming for the moment that X is countably infinite, then the set of possible hat colour assignments may be viewed as Cantor space: the collection of all infinite binary sequences. Two assignments are then equivalent iff their Hamming distance is finite (i.e. they differ in only finitely many coordinates). The equivalence classes are simply the connected components of the Hamming graph based on Cantor space.

3. Of course asking participants to make uncountably many choices, and to remember or store them all, is a tall order. If the existence of a winning strategy for the participants seems too unbelievable to you, then perhaps you may view this explanation as another reason to discredit the Axiom of Choice. On the other hand, if it doesn't sound that surprising to you, just consider the following:

Generalization

Instead of placing just a red or blue hat on each participant's head (a binary choice), I could place on each participant's head a hat on which is printed a finite string of ASCII characters. For example I could write

Notre Dame

on the first hat;

John Deere

on the second;

Real Cowboys Take Baths

on the third;

U8*ttr0+?k

on the fourth; etc. Each participant is allowed to see what is printed on the other hats, but not his own. Each participant must guess what appears on his own hat. If an infinite number of participants guess incorrectly, all are punished by death. The participants again have a winning strategy, easily obtained by modifying the scheme above.

This seems extremely surprising, and again may lead us to question the Axiom of Choice. Note that for any single participant, it seems that the chances of guessing correctly are negligible, or even zero. How then are we to accept that by this scheme, all but a finite number of participants are guaranteed to guess correctly?

Still further comments

The winning strategy for the participants evidently does not require that the set of labels on hats be chosen from a finite set; only that this set of labels be agreed upon beforehand by myself and all participants. On the other hand, the larger this set of legal labels (or the larger the set of participants), then in general, the heavier the reliance on the Axiom of Choice.

I am grateful to my colleague John Hitchcock for introducing me to this problem, and to a variety of related hat problems. It is one of my goals to write a survey on these (that would be in my 'spare time'… ha!)

For more discussion and related hat problems, see *The Mathematics of Coordinated Inference—A Study of Generalized Hat Problems*, Hardin and Taylor, Springer, 2013.