Group Cohomology

Homology and cohomology groups are a very important tool in classifying extensions. The vague term 'extensions' is intended to include various kinds of objects: topological, geometric and algebraic. Here we explain how cohomology groups are useful in classifying central extensions of groups.

1. Modules

Let G be a multiplicative group, and let A be a G-module. This means that A is an additive abelian group and that G acts on A. We denote the image of an element $a \in A$ under an element $g \in G$ by $ga \in A$. To say that G acts on A means that

$$g(a+b) = ga + gb; \qquad (gh)a = g(ha)$$

for all $a, b \in A$; $g, h \in G$.

1.1 Example: Linear Groups

We may take A to be a vector space and let $G = GL(A)$, the group of all invertible linear transformations $A \to A$. Or we may take G to be an arbitrary subgroup of $GL(A)$.

1.2 Example: Trivial Action

Take A to be an arbitrary additive abelian group, and G an arbitrary multiplicative group. The *trivial action* of G on A is defined by

$$ga = a$$

for all $a \in A$, $g \in G$.

2. Definition of Group Cohomology

Let A be a G-module, as in Section 1. Denote by $C^k = C^k(G; A)$ the additive group consisting of all maps $\phi : G^{k+1} \to A$ such that

$$\phi(gg_0, gg_1, \ldots, gg_k) = g\phi(g_0, g_1, \ldots, g_k)$$

for all $g_0, g_1, \ldots, g_k, g \in G$. Such maps are called *$k$-cochains*. The *coboundary* of such a map $\phi \in C^k$ is the $(k+1)$-cochain $\delta\phi \in C^{k+1}$ defined by

$$(\delta\phi)(g_0, g_1, \ldots, g_{k+1}) = \sum_{0 \leqslant i \leqslant k+1} (-1)^i \phi(g_0, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{k+1}).$$

It is easy to check that $\delta\phi$ satisfies the condition above to be a cochain; also that $\delta^2 = 0$ so that we have a cochain complex

$$\cdots \xleftarrow{\delta} C^3 \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \longleftarrow 0.$$

As usual we define the k-th cohomology group of this complex by

$$H^k(G; A) = Z^k(G; A)/B^k(G; A)$$

where $Z^k(G; A)$ is the kernel of $\delta : C^k \to C^{k+1}$ (the additive group of *cocycles*) and $B^k(G; A)$ is the image of $\delta : C^{k-1} \to C^k$ (the additive group of *coboundaries*).

The preceding description of cochains can be abbreviated by a process of *de-homogenization* as we now explain. Every cochain ϕ as above gives rise to a map $f : G^k \to A$ defined by

$$f(g_1, g_2, \ldots, g_k) = \phi(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_k).$$

Conversely we may recover ϕ from f via

$$\phi(g_0, g_1, g_2, \ldots, g_k) = g_0 f(g_0^{-1}g_1, g_1^{-1}g_2, \ldots, g_{k-1}^{-1}g_k).$$

Using this bijection $\phi \leftrightarrow f$ we may identify $C^k(G; A)$ with the additive group of all functions $G^k \to A$. (For those who are familiar with homogeneous coordinates for projective space, this is analogous to the one-to-one correspondence between homogeneous and nonhomogeneous coordinates for points in projective space.) Now we may express the coboundary operator in our new notation as

$$\begin{aligned}(\delta f)(g_1, g_2, \ldots, g_k) = g_1 f(g_2, g_3, \ldots, g_{k+1}) - f(g_1g_2, g_3, \ldots, g_{k+1}) \\ + f(g_1, g_2g_3, g_4, \ldots, g_{k+1}) - \cdots + (-1)^{k+1} f(g_1, g_2, \ldots, g_k).\end{aligned}$$

It is this expression for the coboundary operator, rather than the previous, that we shall use in practice. The only point of giving the previous description in terms of ϕ is to motivate this unusual-looking formula.

2.1 Example: $k = 0$

A 0-cochain is a function $G^0 \to A$. Such a function has no arguments, and so it is really a constant $a \in A$. The coboundary of such a constant is the map

$$\delta a : G \to A, \qquad g \mapsto ga - a.$$

The 0-cocycles are the elements $a \in A$ that are fixed by every element of G.

2.2 Example: $k = 1$

A 1-cochain is a function $f : G \to A$. Its coboundary is the map

$$\delta f : G^2 \to A, \quad (\delta f)(g, h) = gf(h) - f(gh) + f(g).$$

The 1-cocycles are functions $f : G \to A$ satisfying

$$f(gh) = f(g) + gf(h).$$

Such maps are called *crossed homomorphisms* or *derivations.* Note that if the action of G on A is trivial, these are the same as homomorphisms $G \to A$. As a special case one checks directly that every 1-coboundary is a 1-cocycle. Such maps have the form $f(g) = ga - a$ where $a \in A$ is fixed, and are called *principal crossed homomorphisms* or *inner derivations.* They satisfy

$$f(g) + gf(h) = (ga - a) + g(ha - a) = ga - a + gha - ga = gha - a = f(gh)$$

as required.

2.3 Example: $k = 2$

A 2-cochain is a function $f : G^2 \to A$. Its coboundary is the map $\delta f : G^3 \to A$ defined by

$$(\delta f)(g, h, \ell) = gf(h, \ell) - f(gh, \ell) + f(g, h\ell) - f(g, h).$$

Amazingly, such expressions arise naturally in the study of group extensions.

3. Products of Groups

Products (whether direct or semidirect) can be constructed either *internally* or *externally.* To motivate the distinction, consider the probably more familiar situation of direct sums in linear algebra.

3.1 Sums of Vector Spaces

Given two vector spaces U and W (over the same field F) one may construct their *(external) direct sum* which is the new vector space

$$V = U \oplus W = \{(u, w) : u \in U,\ w \in W\}$$

with componentwise addition and scalar multiplication defined by

$$(u, w) + (u', w') = (u+u', w+w'), \qquad c(u, w) = (cu, cw)$$

for all $u, u' \in U$; $w, w' \in W$; $c \in F$. Alternatively, given a vector space V and two subspaces $U, W \leqslant V$, we can realize V as the *(internal) direct sum* of U and W, denoted again as $V = U \oplus W$, provided $U \cap W = \{0\}$ and $U + W = V$. The latter condition means that every vector $v \in V$ can be expressed as $v = u + w$ for some $u \in U$ and $w \in W$; and the preceding condition means that such u and w are uniquely determined by v.

Abstractly there is no distinction between internal and internal direct sums. The difference is only in presentation: namely, does one first define U and W, then construct V as their direct sum? or does one first construct V and then identify a pair of complementary subspaces $U, W \leqslant V$?

Having said this, there is however one subtle distinction between the use of the notation '$\oplus$' for internal and external and internal direct sums, as shown by the following example. We may construct the two-dimensional real vector space $\mathbb{R}^2$ as simply the external direct sum $\mathbb{R} \oplus \mathbb{R}$. However when we view the vector space $\mathbb{R}^2$ as an internal direct sum of two one-dimensional subspaces U and W, these two subspaces should be disjoint. How can we then have $U = W = \mathbb{R}$? This confusion is resolved by observing that U and W are distinct one-dimensional subspaces, namely $U = \{(x, 0) : x \in \mathbb{R}\}$ (the x-axis) and $W = \{(0, y) : y \in \mathbb{R}\}$ (the y-axis). In this case we may rather say $U \cong W \cong \mathbb{R}$ to avoid the notational confusion just observed.

3.2 Direct Products of Groups

We generalize the previous section by taking H and K to be two groups. We assume for now that both H and K are multiplicative. The *(internal) direct product* of H and K is the group

$$G = H \times K = \{(h, k) : h \in H,\ k \in K\}$$

with componentwise multiplication

$$(h,k)(h',k') = (hh',kk').$$

Note that we may identify H with the subgroup $\{(h,1) : h \in H\}$, and identify K with the subgroup $\{(1,k) : k \in K\}$. With this identification, we observe that the subgroups H and K are complementary, i.e. $G = HK = \{hk : h \in H,\ k \in K\}$ (recall the identification of h with $(h,1)$ and k with $(1,k)$) and $H \cap K = 1$ (so that every $g \in G$ can be *uniquely* expressed as $g = hk$) for h and k as above. Moreover these two subgroups are normal and they commute with each other: $hk = kh$ for all $h \in H$ and $k \in K$.

Conversely, given a group G, in order to recognize G as the direct product of two subgroups $H, K \leqslant G$, we require that $G = HK$, $H \cap K = 1$, and H commutes with K (in particular both H and K are normal subgroups). We then write $G = HK = H \times K$, the *(internal) direct product* of H and K.

3.3 Semidirect Products of Groups

Here we generalize the notion of product even further. Let H and K be groups, and suppose that K acts on H. This means that each $k \in K$ determines a map $H \to H$ denoted by $h \mapsto h^k$ such that

$$(h_1h_2)^k = h_1^k h_2^k; \quad h^{k_1k_2} = (h^{k_1})^{k_2}$$

for all $h, h_1, h_2 \in H;\ k, k_1, k_2 \in K$. (Thus we are given not only groups H and K but also a homomorphism $K \to Aut(H)$.) Define the *(external) semidirect product* of H and K as

$$G = H \rtimes K = \{(h,k) : h \in H,\ k \in K\}$$

where the product in G is defined by

$$(h_1,k_1)(h_2,k_2) = (h_1^{k_2}h_2, k_1k_2)$$

for all $h_i \in H,\ k_i \in K$. If you have never done this before, you should check that this actually does define a group; most importantly, this product is associative. Again $\{(h,1) : h \in H\}$ is a subgroup (actually a normal subgroup) which we identify with H; and $\{(1,k) : k \in K\}$ is a subgroup (although not in general normal) which we identify with K. Note that H and K do not typically commute with each other; indeed

$$(1,k)^{-1}(h,1)(1,k) = (h^k,1)$$

so that the original action of K on H which was given, is realized as the action by conjugation in the group G. It is important to realize that the data required to construct the group G includes not only the groups H and K, but also the choice of action of K on H. In particular if one chooses the trivial action, one obtains simply a direct product as a special case.

Reversing our viewpoint, suppose we are given a group G and two subgroups $H, K \leqslant G$ such that H is normal and every element $g \in G$ is uniquely expressible as $g = hk$ where $h \in H,\ k \in K$ (i.e. $G = HK$ with $H \cap K = 1$). Then G is the *(internal) semidirect product* of H and K.

As a special case suppose A is a module for a group K. Then the elements of the semidirect product $A \rtimes K$ can naturally be denoted as matrices

$$\begin{pmatrix} k & 0 \\ a & 1 \end{pmatrix}, \qquad k \in K,\ a \in A$$

with the convention that

$$\begin{pmatrix} k_1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} k_2 & 0 \\ a_2 & 1 \end{pmatrix} = \begin{pmatrix} k_1 k_2 & 0 \\ a_1^{k_2} + a_2 & 1 \end{pmatrix}.$$

Thus for example if $K = GL(A)$ where A is a k-dimensional vector space over a field F, then then $A \rtimes K$ is isomorphic to the group of all invertible $(k+1) \times (k+1)$ matrices over F with last column equal to the transpose of $(0, 0, \ldots, 0, 1)$.

As another example, consider a cyclic group $H = \{1, x, x^2, \ldots, x^{n-1}\}$ of order n, and let $K = \{1, y\}$ be a group of order 2. Then any semidirect product of H by K is either a direct product (in which x commutes with y) or a dihedral group (in which $x^y = y^{-1}xy = x^{-1}$).

4. Group Extensions

A *group extension of A by G* (also called a group extension of G by A) is a short exact sequence of groups

$$1 \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow 1.$$

This means that A can be identified with a subgroup of H in such a way that $H/A \cong G$. We often say simply that the group H is an extension of A by G, if there is no ambiguity regarding the choice of homomorphisms. Such a sequence is called *split* if H has a subgroup complementary to A. (If such a complementary subgroup exists, it would necessarily be isomorphic to G; and then H would be isomorphic to a semidirect product $A \rtimes G$.) Two extensions

$$1 \longrightarrow A \longrightarrow H_1 \longrightarrow G \longrightarrow 1, \qquad\qquad 1 \longrightarrow A \longrightarrow H_2 \longrightarrow G \longrightarrow 1$$

are *equivalent* if there exist isomorphisms α, β, γ which yield a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & H_1 & \longrightarrow & G & \longrightarrow & 1 \\ \downarrow & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \downarrow \\ 1 & \longrightarrow & A & \longrightarrow & H_2 & \longrightarrow & G & \longrightarrow & 1\,. \end{array}$$

(It is only necessary to assume that α and γ are isomorphisms, and that β is a homomorphism, since by the Five Lemma β is forced to also be an isomorphism.) Note that $H_1 \cong H_2$ in the case of equivalent extensions. However, the converse fails: if both H_1 and H_2 are extensions of K by G, then they need not be equivalent; even if $H_1 \cong H_2$, the extensions may not be equivalent.

Group extensions are classified using cohomology. Consider especially the case that A is abelian, and identify A with its image in G. The action of $h \in H$ on A by conjugation only depends on the coset $hA \in H/A \cong G$, so this gives an action of G on A; thus A is a G-module. We may ask, given an action of G on A, for a determination of the equivalence classes of extensions of A by G. These extensions are naturally in one-to-one correspondence with the elements of $H^2 = H^2(G;A)$; and the identity element of this group H^2 corresponds to a split extension (i.e. the semidirect product $A \rtimes G$). Moreover in the special case of a split extension $H = A \rtimes G$, the conjugacy classes of subgroups complementary to A, are naturally in one-to-one correspondence with elements of $H^1 = H^1(G;A)$; and the obvious choice of complementary subgroup (namely G) corresponds with the identity element of H^1.