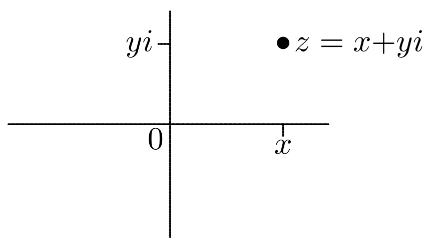
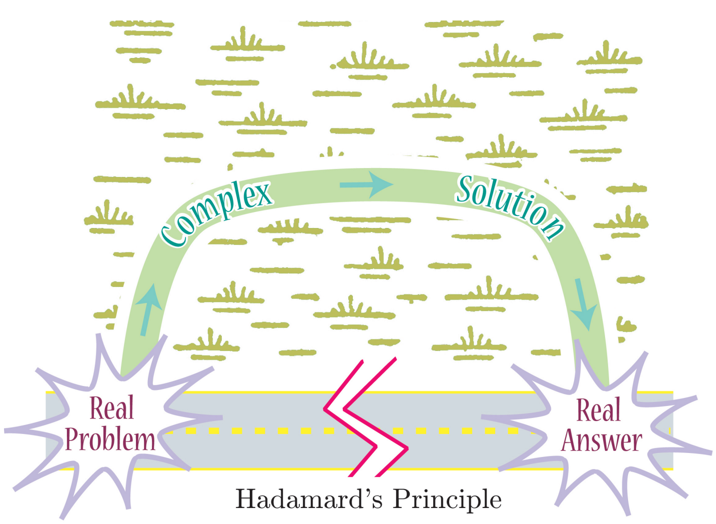
Complex Numbers

The field of complex numbers is the extension $\mathbb{C} \supset \mathbb{R}$ consisting of all expressions $z = x+yi$ where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. We refer to $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ as the *real part* and the *imaginary part* of z, respectively. (*Note:* It is y, *not* yi, which we call the imaginary part of z; thus the real and imaginary parts of z are both real.)



In this context, 'complex' does *not* mean 'complicated'; rather it refers to the fact that z has two parts (the real and the imaginary part), just as a sports complex has more than one part (typically a gym, a pool, a track, etc.) and the vitamin B-complex consists of several separate compounds (vitamins B-1, B-2, B-3, etc.); actually the complex numbers *simplify* our understanding of mathematics and physics. Many properties of real numbers are best understood using complex numbers.



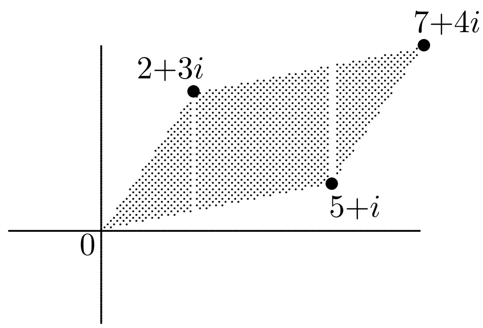
Hadamard's Principle

"The shortest path between two truths in the real domain passes through the complex domain."

Jacques Hadamard, 1865-1963

Soon enough we will encounter examples of Hadamard's Principle.

Complex numbers are added just like vectors in $\mathbb{R}^2$, by adding separately their real and imaginary parts. For example, we show the parallelogram law of addition for the statement $(2{+}3i) + (5{+}i) = 7{+}4i$:



Using the identity

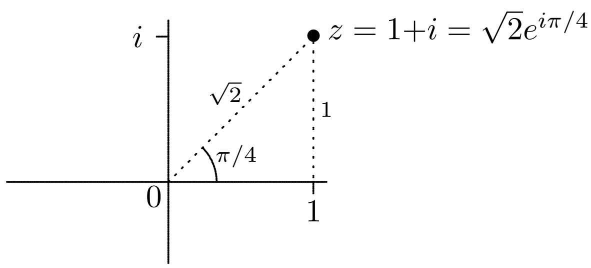
$$e^{i\theta} = \cos\theta + i\sin\theta$$

known as *DeMoivre's Theorem,* complex numbers may be written also in **polar form**; we have $z = x + yi = re^{i\theta}$ where $x = r\cos\theta$ and $y = r\sin\theta$, which allows us to convert between rectangular and polar form. In general, the rectangular coordinates x, y of any complex number are unique; but the polar coordinates r, θ are not. One standard choice is given by $r = \sqrt{x^2 + y^2}$ and $0 \leqslant \theta < 2\pi$ which determines θ uniquely as long as $z \neq 0$; in this case r is the *modulus* of z, denoted $r = |z|$; and θ is the *argument* of z, denoted $\theta = \arg z$. Note that the modulus $|z|$ of a complex number z is just the distance from z to the origin. This coincides with the meaning of $|x|$ for any real number x: the absolute value of x is also the distance from x to the origin.

Example 1. Convert $1{+}i$ to polar form.

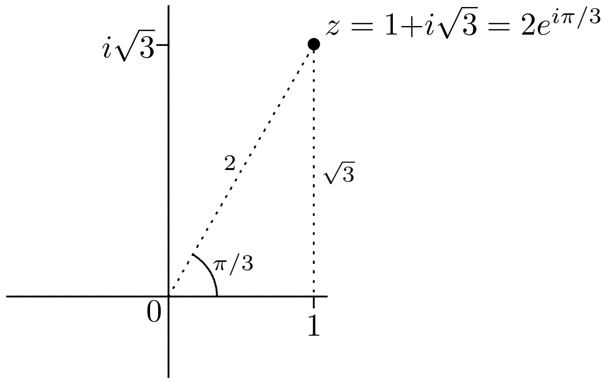
Solution. $r = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\tan\theta = 1/1 = 1$ so $\theta = \arg(1 + i) = \pi/4$; thus

$$1{+}i = \sqrt{2}e^{i\frac{\pi}{4}}.$$



Example 2. Convert $2e^{i\frac{\pi}{3}}$ to rectangular form.

Solution. $2e^{i\frac{\pi}{3}} = 2\cos\frac{\pi}{3} + 2i\sin\frac{\pi}{3} = 1 + i\sqrt{3}$.

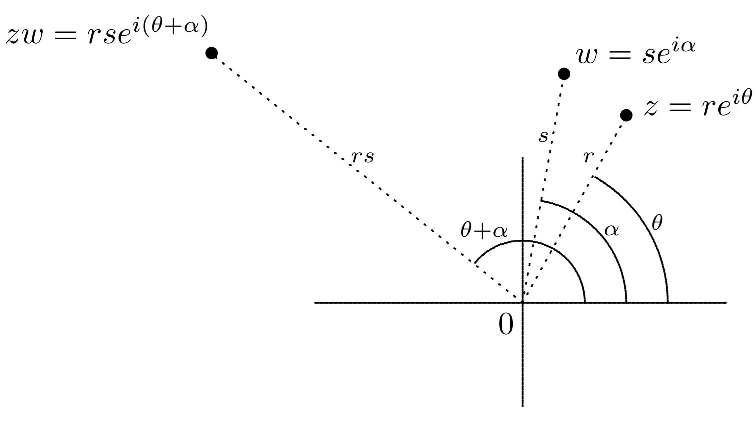


Arithmetic operations with complex numbers may be performed using either their rectangular forms or their polar forms, although rectangular forms are generally more suitable for addition and subtraction; as we shall see, polar forms are often more suitable for multiplication and division, or raising to powers.

Example 3. $(2 + 3i)(5 + i) = 10 + 3(-1) + 2i + 15i = 7 + 17i$.

Example 4. $\frac{2+3i}{5+i} = \frac{2+3i}{5+i} \cdot \frac{5-i}{5-i} = \frac{13+13i}{25+1} = \frac{1}{2} + \frac{i}{2}$.

To provide a geometric interpretation for the multiplication of complex numbers (analogous to the parallelogram law for *addition* of complex numbers), we use the polar form. If $z = re^{i\theta}$ and $w = se^{i\alpha}$, then $zw = rse^{i(\theta+\alpha)}$; thus the magnitude of zw is the product of the magnitudes (or moduli) of z and w; and the argument of zw is the sum of the arguments of z and w.



Example 5. $(1 + i)^7 = \left(\sqrt{2}e^{\pi i/4}\right)^7 = 8\sqrt{2}e^{7\pi i/4} = 8\sqrt{2}\cdot\frac{1}{\sqrt{2}}\left(1-i\right) = 8 - 8i$. This is much faster than using the Binomial Theorem to expand with rectangular coordinates, viz.:

$$(1 + i)^7 \;=\; 1 + 7i - 21 - 35i + 35 + 21i - 7 - i \;=\; 8 - 8i.$$

For every positive integer n, there are n complex roots of the equation $z^n = 1$, called the *n-th roots of unity*. Note that these all have the form

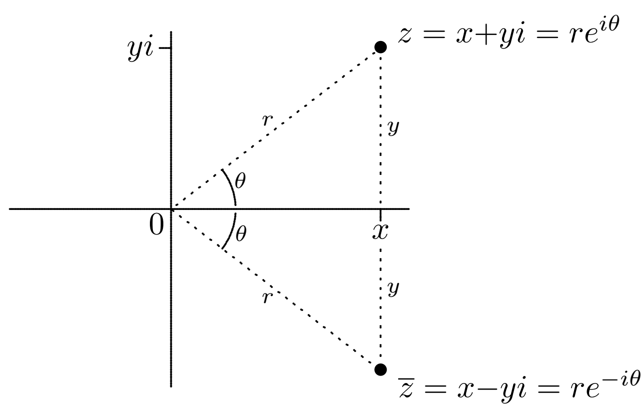
$$\zeta^k = e^{k\pi i/n} = \cos\tfrac{k\pi}{n} + i\sin\tfrac{k\pi}{n}, \qquad k = 0, 1, 2, \ldots, n{-}1$$

where $\zeta = e^{\pi i/n}$. These numbers form the vertices of a regular n-gon inscribed in the unit circle $|z| = 1$ in the complex plane. More generally, every nonzero complex number has exactly n n-th roots, and these roots form the vertices of a regular n-gon. Indeed, let a be any nonzero complex number, and write $a = re^{i\theta}$ in polar form where $r = |a|$; then the roots of $z^n = a$ are

$$r^{1/n}e^{(\theta+k\pi)i/n} = r^{1/n}\cos\tfrac{\theta+k\pi}{n} + ir^{1/n}\sin\tfrac{\theta+k\pi}{n}, \qquad k = 0, 1, 2, \ldots, n{-}1$$

and these form the vertices of a regular n-gon inscribed in the circle $|z| = r^{1/n}$.

The **complex conjugate** of any complex number z, denoted $\overline{z}$, is obtained by reflecting z in the real axis; this preserves the real part and the modulus; and it changes the sign of the imaginary part and the argument:



Observe that for $z = x{+}yi$ we have $z\overline{z} = (x{+}yi)(x{-}yi) = x^2{+}y^2 = |z|^2$; thus $|z| = \sqrt{z\overline{z}}$. Also note that $\overline{z + w} = \overline{z}{+}\overline{w}$ and $\overline{zw} = \overline{z}{\cdot}\overline{w}$ for all $z, w \in \mathbb{C}$. Complex conjugation $z \mapsto \overline{z}$ is an *automorphism* of $\mathbb{C}$: it is bijective, and preserves the field operations. The real and imaginary part of $z = x{+}yi$ are expressible as

$$\mathrm{Re}(z) = \tfrac{1}{2}\big(z + \overline{z}\big); \qquad \mathrm{Im}(z) = \tfrac{1}{2i}\big(z - \overline{z}\big).$$

Functions of the form $f : \mathbb{R} \to \mathbb{R}$ are often understood in terms of their graphs $\{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2$; indeed a function f is often *identified* with its graph. This description is more problematic for functions $f : \mathbb{C} \to \mathbb{C}$, where the graph $\{(z, f(z)) : z \in$

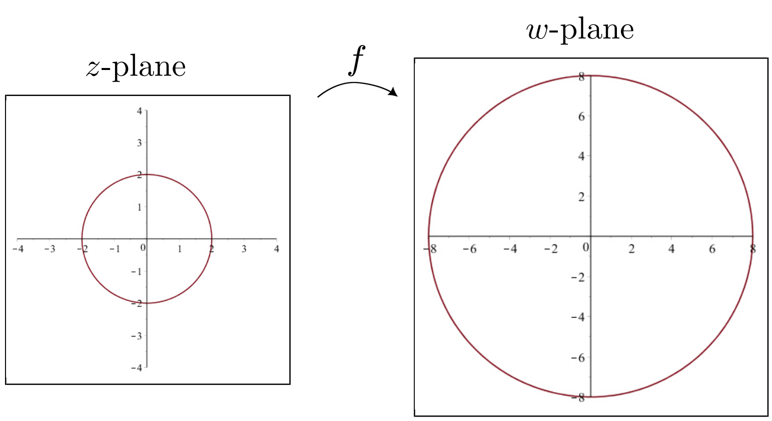
$\mathbb{C}\}$ is typically a *surface* in $\mathbb{C}^2 \simeq \mathbb{R}^4$. Such a graph requires two real coordinates for the domain and another two real coordinates for the range. In order to gain insight into the behaviour of such functions, we often rely on cross sections or projections of the graph, rather than the full graph itself.

For example one may plot $u(x,y) = Re(f(z)) = Re(f(x{+}yi))$ and $v(x,y) = Im(f(z))$ $= Im(f(x{+}yi))$ as two separate surfaces in $\mathbb{R}^3$, then try to put these two pictures together to get an understanding of f itself. Or one may try to illustrate $u(x,y)$ and $v(x,y)$ using their contour plots in the x,y-plane (using a technique studied Calculus III for studying functions of the form $u = u(x,y)$ or $v = v(x,y)$). In our proof of the following result, we will use a different approach: simply consider the image of a circle $|z| = R$ in the z-plane (i.e. the x,y-plane) under the transformation $w = f(z)$.

> **Fundamental Theorem of Algebra.** Every polynomial $f(z) \in \mathbb{C}[z]$ of degree $n \geqslant 1$ has a root (also called a 'zero') $r \in \mathbb{C}$.

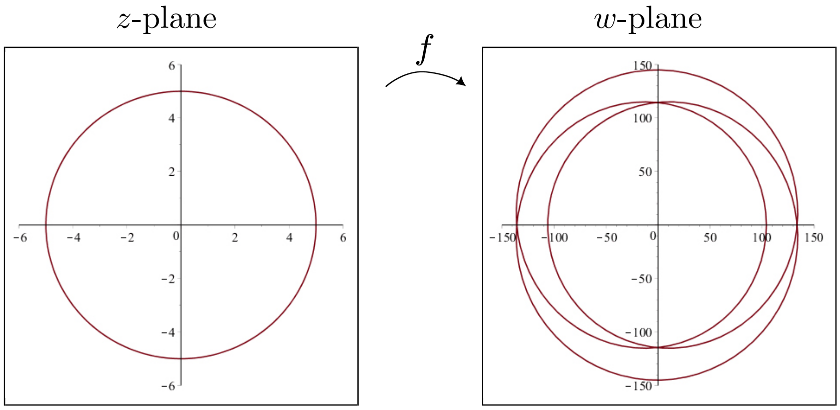
This says that the field of complex numbers is *algebraically closed* and as the name implies, it is a cornerstone of modern algebra. Gauss first proved this result as a graduate student in 1799, although he omitted some of the ideas of the proof. Today several proofs are available, including one often seen in our Math 4230 (Complex Analysis) course. The proof we give here is topological in nature; and although it relies at one point on intuition, the main argument is so accessible that we feel that our readers will forgive our heuristic approach.

To first get an idea of how the proof works, first consider the example the monomial function $f(z) = z^3$. Writing $z = Re^{i\theta}$ we see that $w = f(z) = R^3 e^{i3\theta}$, so that for z-values lying on a fixed circle of radius R centered at 0, corresponding w-values lie on a circle of radius R^3 centered at 0; and as z moves around the circle $|z| = R$ *once* in the counter-clockwise direction, corresponding w-values move counter-clockwise around the circle $|w| = R^3$ *three times* in the counter-clockwise direction. We illustrate with the case $R = 2$:

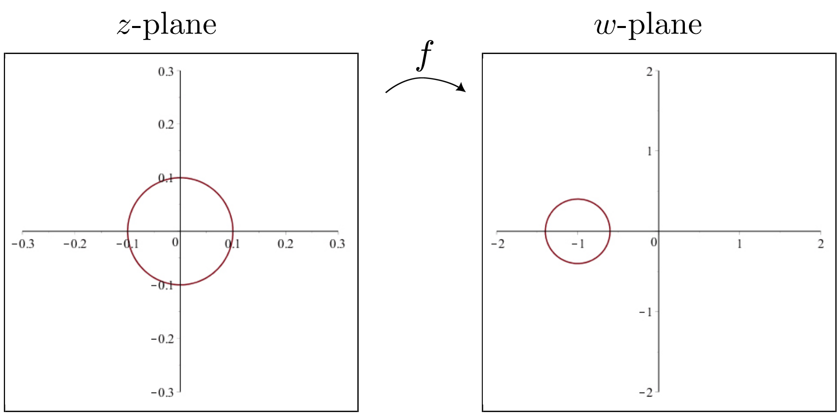


More generally for any monomial of the form $w = f(z) = z^n$, for every time z goes once around the circle $|z| = R$ in the counter-clockwise direction, w goes n times around the circle $|w| = R^n$ in the counter-clockwise direction.

Now consider the more typical example $f(z) = z^3 - 4z - 1$. If $|z| = R$ is large then $f(z)$ is dominated by its leading term z^3 and so $f(z) \approx z^3$. Consider for example the case $R = 5$:

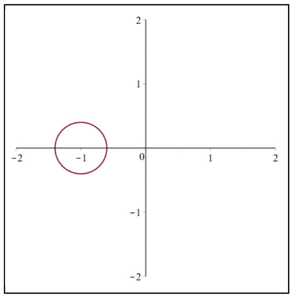


(The reason that the image is not a perfect circle is that $f(z)$ is only *approximately* z^3.) On the other hand if $|z| = R$ is very small, the constant term -1 dominates and $f(z) \approx -1$. Consider the image of the circle $|z| = 0.1$ for example:



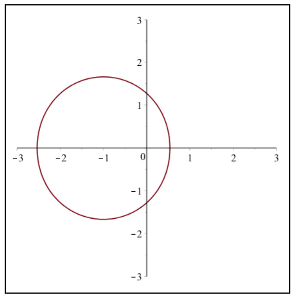
The point is that as $|z| = R$ shrinks from 5 to 0.1, the image curve in the w-plane is forced to pass through the origin. When it does, there must be a z-value with the corresponding magnitude R between 0.1 and 5, such that $f(z) = 0$. What does the image of the circle $|z| = R$ look like in the w-plane for these intermediate values of R? Here are a few examples:

w-plane



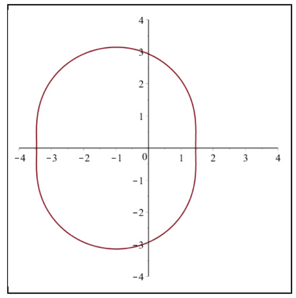
for $|z| = R = 0.1$

w-plane



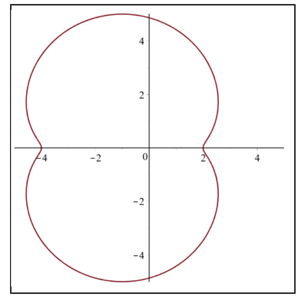
for $|z| = R = 0.4$

w-plane



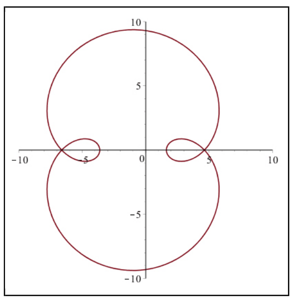
for $|z| = R = 0.7$

w-plane



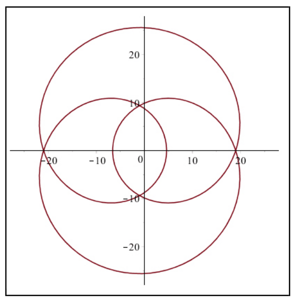
for $|z| = R = 1.0$

w-plane



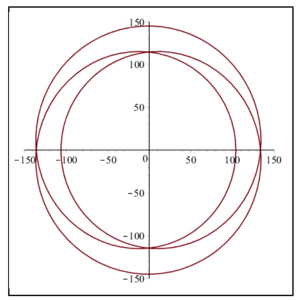
for $|z| = R = 1.5$

w-plane



for $|z| = R = 2.5$

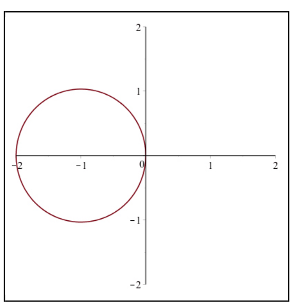
w-plane



for $|z| = R = 5.0$

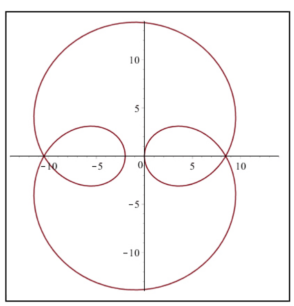
In fact with a little more fine tuning we can find the values of R (correct to three decimal places) for which the curve in the w-plane actually passes through the origin:

w-plane



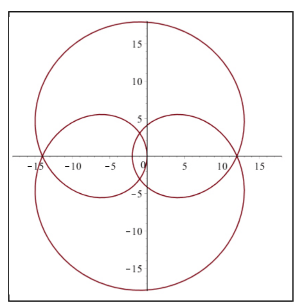
for $|z| = R = 0.254$

w-plane



for $|z| = R = 1.861$

w-plane



for $|z| = R = 2.114$

So in fact $f(z)$ has three zeroes in $\mathbb{C}$, with magnitudes as listed above. (One may verify that $f(z)$ has in fact three real zeroes, namely -1.861, -0.254 and 2.114, correct to three decimal places.)

Finally, here is a proof of the Fundamental Theorem of Algebra: We may suppose $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. (If $f(z)$ is not monic then divide $f(z)$ by its leading coefficient; the resulting polynomial will be monic and it will have the same zeroes as $f(z)$.) We may also assume a_0 is nonzero; otherwise $f(0) = 0$ and we are done. Let M be the maximum of the numbers

$$1,\ 2n|a_0|,\ 2n|a_1|,\ \ldots,\ 2n|a_{n-1}|.$$

Thus $1 \leqslant M \leqslant M^2 \leqslant M^3 \leqslant \cdots$; also $|a_k| \leqslant \frac{M}{2n}$ which implies that

$$|a_k|M^k \leqslant \frac{M^n}{2n} \quad \text{for } k = 0, 1, 2, \ldots, n{-}1.$$

We will show that the approximation $f(z) \approx z^n$ for $|z| = M$ in the following strict sense: if $|z| = M$ then $|f(z) - z^n| \leqslant \frac{M^n}{2}$. The geometric meaning of this formula is that z^n is closer to $f(z)$ than it is to zero. This means that as z goes around the circle $|z| = M$ once counterclockwise, z^n must go around the origin n times counter-clockwise; and since $w = f(z)$ stays close to z^n, it must also go around the origin n times. It is easy to check that the required bound holds:

$$\begin{aligned}
|f(z) - z^n| &= |a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}| \\
&\leqslant |a_0| + |a_1 z| + \cdots + |a_{n-1} z^{n-1}| \\
&= |a_0| + |a_1|M + \cdots + |a_{n-1}|M^{n-1} \\
&\leqslant \tfrac{M^n}{2n} + \tfrac{M^n}{2n} + \cdots + \tfrac{M^n}{2n} \\
&= n \cdot \tfrac{M^n}{2n} \\
&= \tfrac{M^n}{2}
\end{aligned}$$

as claimed.

If $|z| = R$ where R is sufficiently large, say $R = M$ as above, the corresponding values of $w = f(z)$ lie on a curve which 'winds around' the origin. As $|z| = R$ shrinks, values of $w = f(z)$ approach the constant term a_0 which is nonzero; and in the limit when $z = 0$, the corresponding w-value is just the point $f(0) = a_0$ which does not wind around the origin at all. By continuity, there must be an intermediate value of R between 0 and M, for which f maps the circle $|z| = R$ to a curve in the w-plane which passes through the origin; in other words $f(z) = 0$ for some $z \in \mathbb{C}$ such that $|z| = R$. The continuity argument, similar to the Intermediate Value Theorem learned in Calculus I, can be made precise using topological arguments; but our heuristic version of the proof should suffice for most casual readers.

If $f(z) \in \mathbb{C}$ has a zero $r \in \mathbb{C}$ then we can factor $f(z) = (z - r)g(z)$ where $\deg g(z) = n{-}1$. Assuming $n \geqslant 2$, we may repeat the argument to obtain a linear factor of $g(z)$; continuing this process we eventually factor $f(z)$ into n linear factors:

$$f(z) = a(z - r_1)(z - r_2)\cdots(z - r_n), \quad a, r_1, r_2, \ldots, r_n \in \mathbb{C}.$$

Corollary. Every real polynomial $f(X) \in \mathbb{R}[X]$ factors into linear and quadratic factors.

Proof. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ where $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$. (Again there is no loss of generality in assuming $f(X)$ is monic.) By the Fundamental Theorem of Algebra there exist $r_1, r_2, \dots, r_n \in \mathbb{C}$ such that

$$f(X) = (X - r_1)(X - r_2)\cdots(X - r_n).$$

Since the coefficients of $f(X)$ are real, we have $\overline{f(X)} = f(X)$ and so using the properties of complex conjugation,

$$(X - \overline{r_1})(X - \overline{r_2})\cdots(X - \overline{r_n}) = (X - r_1)(X - r_2)\cdots(X - r_n).$$

This means that the complex numbers $\overline{r_1}, \overline{r_2}, \dots, \overline{r_n}$ are the same as $r_1, r_2, \dots, r_n$ listed in some order. In particular the complex zeroes $r_1, r_2, \dots, r_n$ of $f(z)$ are of two possible types:

(i) real zeroes satisfying $\overline{r_k} = r_k$; and
(ii) pairs of non-real zeroes $\{r_k, \overline{r_k}\}$ where $r_k \notin \mathbb{R}$.

After reordering the zeroes in a different order if necessary we may assume that the non-real zeroes of $f(X)$ are $r_1, \overline{r_1}, \dots, r_m, \overline{r_m}$ and the real zeroes are $r_{2m+1}, \dots, r_n$ so that

$$f(X) = (X - r_1)(X - \overline{r_1})\cdots(X - r_m)(X - \overline{r_m})(X - r_{2m+1})\cdots(X - r_n).$$

For $k = 1, 2, \dots, m$ we have

$$(X - r_k)(X - \overline{r_k}) = X^2 - (r_k + \overline{r_k})X + r_k\overline{r_k} = X^2 - 2Re(r_k)X + |r_k|^2 \in \mathbb{R}[X]$$

and clearly the remaining factors $X - r_{2m+1}, \dots, X - r_n$ are also real. □

The latter result is a good example of Hadamard's Principle: We understand that every real polynomial factors into linear and irreducible quadratic factors over $\mathbb{R}$, since these factors correspond to the real zeroes and the pairs of non-real zeroes in $\mathbb{C}$, respectively. This is a result about the real numbers, which we have no hope of understanding without the complex numbers.

Another instance of Hadamard's Principle is the following: Let $A, B \in \mathbb{R}$. Then

$$\begin{aligned}\cos(A{+}B) + i\sin(A{+}B) &= e^{i(A+B)} \\ &= e^{iA}e^{iB} \\ &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin A\sin B.\end{aligned}$$

Comparing real and imaginary parts on both sides, we conclude that

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B, \\ \sin(A+B) &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

These formulas arise frequently in trigonometry over the real numbers, and are useful in Calculus I; and although there are several ways to derive these formulas, there is no proof shorter than the one we have given using complex numbers.

Here is yet another demonstration of Hadamard's Principle arising in number theory:

Example 6. Express the number 6161 as a sum of two integer squares.

Solution. Note that $6161 = 61 \cdot 101$ where it is possible by mere inspection to express each factor as a sum of two squares: $61 = 5^2 + 6^2$ and $101 = 10^2 + 1^2$. Since $61 = |z|^2$ and $101 = |w|^2$ where $z = 5 + 6i$ and $w = 10 + i$, we have $6161 = |z|^2|w|^2 = |zw|^2$ where

$$zw = (5 + 6i)(10 + i) = 44 + 65i.$$

This gives a solution $6161 = |zw|^2 = 44^2 + 65^2$.