Fibonacci Numbers II (Handout January 25, 2013)

We investigate more closely the Fibonacci sequence. We obtain a new proof of a result we have already seen—the closed formula for F_n.

As before, we define the Fibonacci sequence recursively by

$$F_n = \begin{cases} 1, & \text{if } n = 0 \text{ or } 1; \\ F_{n-1} + F_{n-2}, & \text{if } n \geqslant 2. \end{cases}$$

It is convenient to extrapolate backwards using the recurrence formula to obtain $F_{-1} = 0$, $F_{-2} = 1$, $F_{-3} = -1$, $F_{-4} = 2$, etc. From the recurrence relation for F_n, we deduce the matrix equation

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

valid for all $n \geqslant 0$. If we denote $v_n = \left[\begin{smallmatrix} F_n \\ F_{n-1} \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then the latter recurrence may be expressed succinctly as

$$v_{n+1} = Av_n \quad \text{for all } n \geqslant 0.$$

This clearly yields

$$v_n = Av_{n-1} = A^2 v_{n-2} = \cdots = A^n v_0.$$

Fortunately, it is easy to calculate A^n by first diagonalizing. Since A is a symmetric real matrix, it has two real eigenvalues which are the roots of the characteristic polynomial

$$\det\bigl(xI - A\bigr) = \begin{vmatrix} x-1 & -1 \\ -1 & x \end{vmatrix} = x^2 - x - 1.$$

As we have seen, the roots of this polynomial are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

For each eigenvalue $\lambda \in \{\alpha, \beta\}$ we find a corresponding eigenvector v by solving the linear system $Av = \lambda v$. We obtain eigenvectors $\left[\begin{smallmatrix} \alpha \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \beta \\ 1 \end{smallmatrix}\right]$ for the eigenvalues α and β, respectively. The change-of-basis matrix $B = \left[\begin{smallmatrix} \alpha & \beta \\ 1 & 1 \end{smallmatrix}\right]$ is obtained by taking eigenvectors as its columns. We compute the inverse matrix $B^{-1} = \frac{1}{\sqrt{5}}\left[\begin{smallmatrix} 1 & -\beta \\ -1 & \alpha \end{smallmatrix}\right]$. The diagonal form for A

is the diagonal matrix of eigenvalues $D = \left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right]$. We have successfully diagonalized A as $B^{-1}AB = D$, i.e.

$$A = BDB^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix}.$$

This expression allows us to explicitly determine powers of A using $B^{-1}B = I$, as in the example

$$A^3 = \left(BDB^{-1}\right)\left(BDB^{-1}\right)\left(BDB^{-1}\right) = BDDDB^{-1} = BD^3B^{-1}.$$

The same reasoning shows that in general,

$$\begin{aligned} A^n = BD^nB^{-1} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & -\beta \\ -1 & \alpha \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n+1}-\beta^{n+1} & \alpha\beta^{n+1}-\alpha^{n+1}\beta \\ \alpha^n-\beta^n & \alpha\beta^n-\alpha^n\beta \end{bmatrix} \end{aligned}$$

for all $n \geqslant 0$. Observe that $v_0 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and so $v_n = A^n \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, which is just the first column of A. This yields the identity

$$F_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}},$$

our second proof of this fact. One of the lessons here is that two proofs are better than one (at least from a pedagogical viewpoint).

Homework #1 Due Friday, February 1, 2013

Please refer to the course syllabus for general instructions on completing and submitting homework.

Prove that the Fibonacci numbers satisfy the identity $F_n^2 = F_{n+1}F_{n-1} + (-1)^n$ for all $n \geqslant 1$

(a) by induction, using the recurrence relation $F_{n+1} = F_n + F_{n-1}$; and
(b) directly, using a closed formula for F_n.

Note that I am asking you to give two proofs of one result.
