Fibonacci Numbers (Handout January, 2016)

This document summarizes the introduction to Fibonacci numbers presented during our first week of semester. This quick introduction is intended to highlight some of the upcoming themes of our course (as these themes are less apparent in the first couple of chapters of the textbook).

The Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... is defined[1] recursively by

$$F_n = \begin{cases} 1, & \text{if } n = 1 \text{ or } 2; \\ F_{n-1} + F_{n-2}, & \text{if } n \geqslant 3. \end{cases}$$

We proceed to describe one of the many settings in which this sequence unexpectedly appears.

A *bit* (abbreviation for '*binary digit*') is defined to be a symbol '0' or '1'. (Note that we consider bits a symbols, like letters, not as numbers.) By a *bitstring* (or *binary string*), we mean a finite sequence of bits. For each $n \geqslant 0$, there are exactly 2^n bitstrings of *length* n, where the length of a bitstring is defined to be the number of bits in the string. We include the case $n = 0$, which yields the *null string* '' of length zero. Usually we omit the quotation marks, abbreviating '01101' as 01101 for example; but for the null string, clearly such abbreviation won't work. Here we list explicitly all bitstrings of length $\leqslant 3$:

n	bitstrings of length n
0	''
1	0, 1
2	00, 01, 10, 11
3	000, 001, 010, 011, 100, 101, 110, 111

Our problem is to determine the number of bitstrings of length n having no two consecutive 1's. Let us call such a bitstring a 11-free bitstring, and denote by a_n the number of 11-free bitstrings of length n. The following table shows that the first few terms in the sequence a_n look remarkably like Fibonacci numbers:

[1] This sequence appears in the exercises to Chapter 8 of the textbook.

n	11-free bitstrings of length n	a_n = number of such bitstrings	b_n = no. ending in '1'	c_n = no. *not* ending in '1'
0	''	1	0	1
1	0, 1	2	1	1
2	00, 01, 10	3	1	2
3	000, 001, 010, 100, 101	5	2	3
4	0000, 0001, 0010, 0100, 1000, 0101, 1001, 1010	8	3	5

For future reference, I have also indicated how many such bitstrings end (or do not end) in '1': there are b_n and c_n such bitstrings respectively, so that $a_n = b_n + c_n$. Based on this limited evidence, we conjecture (i.e. guess) that $a_n = F_{n+2}$, $b_n = F_n$ and $c_n = F_{n+1}$ for all $n \geqslant 0$. Let us actually prove this fact:

Theorem. *For all $n \geqslant 0$, we have $a_n = F_{n+2}$.*

Proof. Every 11-free bitstring of length $n+1$ *not* ending in '1' (call it w) has the form $w = w'0$ where w' is an arbitrary 11-free bitstring of length n; this says that

$$c_{n+1} = a_n = b_n + c_n.$$

Every 11-free bitstring w of length $n+1$ *ending in* '1' has the form $w = w'1$ where w' is necessarily a 11-free bitstring of length n *not* ending in '1'; this says that

$$b_{n+1} = c_n.$$

Substituting this into the previous formula gives $c_{n+1} = c_n + c_{n-1}$ for $n \geqslant 1$, which is the same recurrence formula as that characterizing the Fibonacci sequence. Now it is easy to complete the formal proof using mathematical induction:

The formula $a_n = F_{n+2}$ holds for $n \leqslant 1$ as we see from our table. Now suppose that $n \geqslant 2$; we must show that $a_n = F_n$, assuming that $a_k = F_{k+2}$ for all $k \in \{0, 1, \ldots, n-1\}$. Using this assumption and our recurrence formula for a_n,

$$a_n = a_{n-1} + a_{n-2} = F_{n+1} + F_n = F_{n+2}$$

where we have also used the recurrence formula for the Fibonacci sequence. The result follows by mathematical induction. $\square$

Next, we demonstrate that it is possible to obtain a direct formula for F_n (or for a_n). Such a formula can be used to evaluate F_n without having to evaluate *all* of the previous terms $F_0, F_1, F_2, F_3, \ldots, F_{n-1}$ first. We accomplish this using a method described in Chapter 8 of the textbook, where we introduce the *generating function* of the sequence F_n as

$$\begin{aligned} F(x) &= \sum_{n\geqslant 0} F_n x^n = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \cdots \\ &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots . \end{aligned}$$

It is important to note that this expression is purely symbolic. Here x is a symbol, not a number; and $F(x)$ is also purely symbolic. In particular, calling $F(x)$ a function is a misnomer. We never evaluate $F(a)$ for any number a; and so convergence of the power series is never an issue. For this reason, none of what you learned about power series in Calculus II is needed here.

One of the most important power series is the *geometric series*

$$\frac{1}{1-x} = \sum_{n\geqslant 0} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

The identity is verified by cross-multiplying:

$$\begin{aligned} (1-x)(1+x+x^2+x^3+x^4+\cdots) &= 1 - x + x - x^2 + x^2 - x^3 + x^3 - \cdots \\ &= 1. \end{aligned}$$

Here, as always, x and the series expansion for $\frac{1}{1-x}$, are purely symbolic. (In Calculus II you would have been taught that the series is only valid when the series converges, i.e. for $|x| < 1$; but in our context it is inappropriate to require such a caveat since x is not a number, and in particular $|x|$ has no meaning.)

To obtain a closed-form expression for $F(x)$, use the identity $F_n = F_{n-1} + F_{n-2}$, valid for all $n \geqslant 2$, thus:

$$\begin{aligned} F(x) &= \sum_{n\geqslant 0} F_n x^n = x + \sum_{n\geqslant 2} F_n x^n; \\ F(x) - x &= \sum_{n\geqslant 2} F_n x^n \\ &= \sum_{n\geqslant 2} (F_{n-1} + F_{n-2}) x^n \\ &= \sum_{n\geqslant 2} F_{n-1} x^n + \sum_{n\geqslant 2} F_{n-2} x^n \\ &= x\big(F_1 x + F_2 x^2 + F_3 x^3 + \cdots\big) \;+\; x^2\big(F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots\big) \\ &= xF(x) + x^2 F(x). \end{aligned}$$

Thus $(1-x-x^2)F(x)=x$, i.e.

$$F(x)=\frac{x}{1-x-x^2}\,.$$

In order to decompose $F(x)$ into two terms with linear (rather than quadratic) denominators, we first factor the denominator as

$$1-x-x^2=(1-\alpha x)(1-\beta x)$$

where α and β are the reciprocal roots of the quadratic polynomial (i.e. $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the actual roots of $1-x-x^2$). In particular for $\frac{1}{\alpha}$ to be a root, we require $1-\frac{1}{\alpha}-\frac{1}{\alpha^2}=0$ and so $\alpha^2-\alpha-1=0$, and similarly $\beta^2-\beta-1=0$. Thus

$$\alpha,\beta=\frac{1\pm\sqrt{5}}{2}\,.$$

It doesn't matter which root has the '+' sign and which has the '$-$' sign; we might as well take

$$\alpha=\frac{1+\sqrt{5}}{2}=1.618\ldots;\qquad \beta=\frac{1-\sqrt{5}}{2}=-0.618\ldots.$$

For future reference, observe that $\alpha-\beta=\sqrt{5}$. The decimal approximations for α and β are not strictly needed here, but they are shown to satisfy our curiosity. Note that α is the famous irrational number known as the *golden ratio*. Next, as promised, we split $F(x)$ into two terms as

$$F(x)=\frac{x}{1-x-x^2}=\frac{x}{(1-\alpha x)(1-\beta x)}=\frac{A}{1-\alpha x}+\frac{B}{1-\beta x}$$

where A and B are constants. This decomposition, known as the *partial fraction decomposition of* $F(x)$, is often introduced in Calculus II as a technique for integrating rational functions. The fact that such constants A,B exist is a result in linear algebra; and while we do not require any knowledge of calculus, we do require you to know some linear algebra. In particular the constants A and B are found by solving two linear equations in two unknowns. Start by multiplying both sides of our formula for $F(x)$ by $1-x-x^2=(1-\alpha x)(1-\beta x)$ and cancelling factors where possible to obtain

$$x=(1-x-x^2)F(x)=(1-\alpha x)(1-\beta x)F(x)=(1-\beta x)A+(1-\alpha x)B.$$

This is an identity of polynomials. Evaluating at $x=\frac{1}{\alpha}$ (so that the last term vanishes) yields

$$\tfrac{1}{\alpha}=\big(1-\tfrac{\beta}{\alpha}\big)A=\tfrac{\alpha-\beta}{\alpha}A=\tfrac{\sqrt{5}}{\alpha}A\,,$$

so that $A = \frac{1}{\sqrt{5}}$; and evaluating similarly at $x = \frac{1}{\beta}$ yields

$$\tfrac{1}{\beta} = \left(1 - \tfrac{\alpha}{\beta}\right)B = \tfrac{\beta-\alpha}{\beta}B = -\tfrac{\sqrt{5}}{\beta}B$$

so that $B = -\frac{1}{\sqrt{5}}$. This gives the partial fraction decomposition

$$F(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\Big(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\Big)\,.$$

Using our geometric series expansion, we obtain

$$\begin{aligned} F(x) &= \tfrac{1}{\sqrt{5}}\Big(1 + \alpha x + \alpha^2 x^2 + \alpha^3 x^3 + \cdots\Big) - \tfrac{1}{\sqrt{5}}\Big(1 + \beta x + \beta^2 x^2 + \beta^3 x^3 + \cdots\Big) \\ &= \sum_{n \geqslant 0} \tfrac{\alpha^n - \beta^n}{\sqrt{5}} x^n. \end{aligned}$$

Our closed form expression for the n-th Fibonacci number is obtained by simply reading off the coefficient of x^n:

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}\,.$$

Note that for large values of n, the value of β^n tends to zero since $|\beta| < 1$. A consequence of this is that

$$F_n \approx \frac{\alpha^n}{\sqrt{5}};$$

in fact we may obtain F_n from $\frac{\alpha^n}{\sqrt{5}}$ by simply rounding off to the nearest integer. A consequence of this formula is the fact that F_n grows at an exponential rate, with the golden ratio α as the base of the exponential function.

The accompanying MAPLE worksheet demonstrates several aspects of our example. We recursively determine the first 100 (or 101) terms in the Fibonacci sequence. We then compare with the power series expansion of $F(x)$. We also test our closed formula for F_n (both the exact formula, and the approximation $\frac{\alpha^n}{\sqrt{5}}$. These computations would not be possible using a typical hand-held calculator; but MAPLE implements arbitrary-precision integer arithmetic which makes these computations possible.