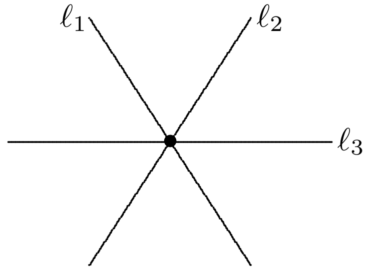
Equiangular Lines

(Handout April 25, 2007)

Consider a set $\{\ell_1, \ell_2, \ldots, \ell_k\}$ consisting of distinct lines passing through a common point, say the origin, in $\mathbb{R}^n$. We call this a set of *equiangular lines* if the angle between ℓ_i and ℓ_j is the same for all choices of $i \neq j$. (Note that the angle between two intersecting lines is always $\leq 90°$.) The problem is: for a given dimension n, what is the maximum size k for a set of equiangular lines? In $\mathbb{R}^2$, it is not hard to find three such lines:



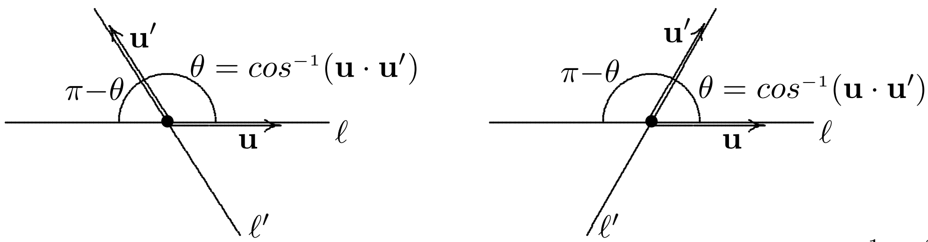
We might believe that 3 is the maximum possible number of equiangular lines in $\mathbb{R}^2$, but how would we prove this?

In class we demonstrated the existence of six equiangular lines, formed by joining pairs of opposite vertices of the icosahedron:

In this example, the common angle between the lines is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \doteq 63.4°$. Once again, the question arises as to whether 6 is the largest possible size for a set of equiangular lines in $\mathbb{R}^3$. In higher dimensions this question is of great interest to some people in graph theory and combinatorics, because of its relationship to other structures. You may recognize that the problem makes sense for any n. Indeed, if $\mathbf{u}$ and $\mathbf{u}'$ are unit vectors in $\mathbb{R}^n$, then $\mathbf{u} \cdot \mathbf{u}' = \|\mathbf{u}\|\,\|\mathbf{u}'\| \cos\theta = \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{u}'$. (Although $\mathbf{u}$ and $\mathbf{u}'$ are in Euclidean space of possibly large dimension, nevertheless $\mathbf{u}$ and $\mathbf{u}'$ lie in a plane, and the angle has the usual geometric meaning within this plane.) The angle between the lines spanned by $\mathbf{u}$ and $\mathbf{u}'$ is either θ or $\pi-\theta$, whichever is acute (or a right angle). Since $\cos(\pi-\theta) = -\cos\theta$, we have

$$\cos^2(\pi-\theta) = \cos^2\theta = (\mathbf{u}\cdot\mathbf{u}')^2$$

an expression which depends only on the angle between the two lines (which is either θ or $\pi-\theta$).



The assertion that the maximum number of equiangular lines in $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$ is 1, 3 or 6 respectively, is a consequence of the following.

Theorem 1. A set of equiangular lines in $\mathbb{R}^n$ has size at most $\binom{n+1}{2}$.

This theorem was reported by Lemmens and Seidel (1973) where the result is attributed to Gerzon. Here we present a much simpler proof, due to Koornwinder (1976), using polynomials.

We first point out that the appearance of the numbers $1, 3, 6, \ldots$ in this context is explained by the following. A *homogeneous polynomial of degree d* is by definition a polynomial all of whose terms have degree d. Homogeneous polynomials of degree 2 have the form

$$f(X) = a_1X^2;$$
$$f(X,Y) = a_1X^2 + a_2XY + a_3Y^2;$$
$$f(X,Y,Z) = a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5XZ + a_6YZ$$

for 1, 2 or 3 variables respectively. This says that the vector space consisting of all homogeneous polynomials of degree 2 in 1, 2 or 3 variables has dimension 1, 3 or 6 respectively. Later we will give a general formula for the dimension of the vector space of all polynomials, or all homogeneous polynomials, of each degree. First we indicate how to prove Theorem 1, at least in the case $n = 3$. (The general case is no harder, using the formula for the dimension of the vector space of all homogeneous polynomials that we will give later.)

Suppose $\{\ell_1, \ell_2, \ldots, \ell_k\}$ is a set of equiangular lines in $\mathbb{R}^3$. Choose a unit vector $\mathbf{u}_i$ in the direction of each ℓ_i. Thus $\mathbf{u}_i\cdot\mathbf{u}_i = 1$ and $\mathbf{u}_i\cdot\mathbf{u}_j = \pm\alpha$ for all $i \neq j$, where $\alpha = \cos\theta$ and θ is the common angle of the set of equiangular lines. For each i, define

$$f_i(X,Y,Z) \;=\; \big(\mathbf{u}_i\cdot(X,Y,Z)\big)^2 - \alpha^2\|(X,Y,Z)\|^2\,,$$

which is a homogeneous polynomial of degree 2 in X,Y,Z. (Note: $(X,Y,Z) = X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ are two notations for the vector in $\mathbb{R}^3$ from the origin to the point (X,Y,Z).) Note that

$$\begin{aligned} f_i(\mathbf{u}_i) \;&=\; (\mathbf{u}_i\cdot\mathbf{u}_i)^2 - \alpha^2\|\mathbf{u}_i\|^2 = 1-\alpha^2 = \sin^2\theta \neq 0;\\ f_j(\mathbf{u}_i) \;&=\; (\mathbf{u}_i\cdot\mathbf{u}_j)^2 - \alpha^2\|\mathbf{u}_j\|^2 = \alpha^2-\alpha^2 = 0 \end{aligned}$$

whenever $i \neq j$. Now it is easy to see that $f_1, f_2, \ldots, f_k$ are linearly independent. Suppose that

$$c_1 f_1(X,Y,Z) + c_2 f_2(X,Y,Z) + \cdots + c_k f_k(X,Y,Z) = 0.$$

By definition, this means that every coefficient on the left hand side is zero. Evaluating both sides at $\mathbf{u}_i$, the only term that survives is the i-th term, and we get $c_i(1-\alpha^2) = 0$. Thus $c_i = 0$ for $i = 1,2,\ldots,k$. We have shown that $f_1, f_2, \ldots, f_k$ are linearly independent. However, these polynomials all belong to the space of homogeneous polynomials of degree 2 in X,Y,Z:

$$f_1, f_2, \ldots, f_k \;\in\; \big\{a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5XZ + a_6YZ \;:\; a_1,\ldots,a_6 \in \mathbb{R}\big\}\,.$$

Since the latter vector space has dimension 6, any linearly independent subset has size at most 6. Thus $k \le 6$. □

The above theorem leaves several questions unanswered. For example, it does not say there exist $\binom{n+1}{2}$ equiangular lines in $\mathbb{R}^n$; it only says there cannot be more than this many equiangular lines. For example in $\mathbb{R}^4$, the maximum size of a set of equiangular lines is 6, as proven by Haantjes (1948). This is less than the upper bound of $\binom{4+1}{2} = 10$. Moreover, the theorem does *not* say that there is only one possible arrangement of three equiangular lines in $\mathbb{R}^2$, or of six equiangular lines in $\mathbb{R}^3$.

Finally, let us explain the formula for the upper bound appearing in Theorem 1. This arises from the following more general result:

Lemma 2. The number of monomials of degree d in $X_0, X_1, X_2, \ldots, X_n$ equals $\binom{n+d}{n}$. This also equals the number of monomials of degree *at most* d in $X_1, X_2, \ldots, X_n$.

Recall that the *binomial coefficient* $\binom{n}{k}$ (‘*n choose k*’), defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1}\,,$$

equals the number of ways of choosing k objects from n objects. Thus the number of vectors of length 5 having two 0’s and three 1’s equals $\binom{5}{2} = 10$ (since each such vector corresponds to a choice of 2 out of 5 positions to place the 0's); it also equals $\binom{5}{3} = 10$ (since one may alternatively choose of 3 out of 5 positions to place the 1's). More generally, it is easy to see that $\binom{n}{k} = \binom{n}{n-k}$.

The one-to-one correspondences illustrated by:

1	$\longleftrightarrow$	X^3	$\longleftrightarrow$	□□□■■
Y	$\longleftrightarrow$	X^2Y	$\longleftrightarrow$	□□■□■
Y^2	$\longleftrightarrow$	XY^2	$\longleftrightarrow$	□■□□■
Y^3	$\longleftrightarrow$	Y^3	$\longleftrightarrow$	■□□□■
Z	$\longleftrightarrow$	X^2Z	$\longleftrightarrow$	□□■■□
YZ	$\longleftrightarrow$	XYZ	$\longleftrightarrow$	□■□■□
Y^2Z	$\longleftrightarrow$	Y^2Z	$\longleftrightarrow$	■□□■□
Z^2	$\longleftrightarrow$	XZ^2	$\longleftrightarrow$	□■■□□
YZ^2	$\longleftrightarrow$	YZ^2	$\longleftrightarrow$	■□■□□
Z^3	$\longleftrightarrow$	Z^3	$\longleftrightarrow$	■■□□□

show that there are precisely $\binom{2+3}{2}$ monomials of degree 3 in X, Y and Z, and the same number of monomials of degree *at most* 3 in Y and Z. The first correspondence is defined by $Y^jZ^k \leftrightarrow X^iY^jZ^k$ where $i = 3-j-k$. Each monomial $X^iY^jZ^k$ of degree 3 corresponds to a sequence of five squares consisting of i hollow squares, then a solid square, followed by j hollow squares and another solid squares, and finally by k hollow squares.

A proof of Lemma 2 in the general case follows from the following more general one-to-one correspondences, which generalize the example above:

$$\left\{\begin{array}{c}\text{monomials}\\ X_1^{e_1}X_2^{e^2}\cdots X_n^{e_n},\\ e_1+e_2+\cdots+e_n \le d\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{monomials}\\ X_0^{e_0}X_1^{e_1}X_2^{e_2}\cdots X_n^{e_n},\\ e_0+e_1+e_2+\\ \cdots+e_n = d\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{sequences of}\\ n\text{ hollow}\\ \text{squares `}\square\text{'}\\ \text{and } d \text{ solid}\\ \text{squares `}\blacksquare\text{'}\end{array}\right\}$$

given by $X_1^{e_1}X_2^{e_2}\cdots X_n^{e_n} \leftrightarrow X_0^{e_0}X_1^{e_1}X_2^{e_2}\cdots X_n^{e_n}$ for any nonnegative integers $e_1, e_2, \ldots, e_n$ whose sum is d; these both correspond to the sequence of $n+d$ squares consisting of e_0 hollow squares, then a solid square, then e_1 hollow squares, then another solid square, ..., then a solid square, then e_n hollow squares. The number of such sequences of squares is $\binom{n+d}{d}$. □

By Lemma 2 we can now see the dimension of the vector space of polynomials of degree $\le d$ in an arbitrary number of variables. We can also see the dimension of a space of homogeneous polynomials in any number of variables. Every polynomial can be split up into its homogeneous parts; for example, we may write

$$\begin{aligned} f(X,Y,Z) &= 13X^3 + 4X^2 + X - 17Y^3 + 11Y^2 + 2XY^2 - 3XY - 5 \\ &= (-5) + (X) + (4X^2 - 3XY + 11Y^2) + (13X^3 + 2XY^2 - 17Y^3)\,, \end{aligned}$$

where the parts in parentheses are the homogeneous parts of degree 0, 1, 2 and 3, respectively.

Theorem 3. (i) The vector space consisting of polynomials of degree $\leq d$ in n indeterminates, has dimension $\binom{n+d}{d}$.
(ii) The subspace consisting of homogeneous polynomials of degree d in n indeterminates, has dimension $\binom{n-1+d}{d}$.

The proof of Theorem 3 follows directly from Lemma 2; however, note that we replaced $n+1$ by n to obtain part (ii) from Lemma 2.