Dirichlet Series

You should be very familiar with the following fact from Calculus II.

Theorem A. *The harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ *diverges.*

This was proved using the Integral Test: $\int_1^\infty \frac{1}{t}\,dt$ diverges since $\int_1^c \frac{1}{t}\,dt = \ln(c) \to \infty$ as $c \to \infty$.

Another proof that the series diverges is to observe that

$$\begin{aligned}
1 &\geqslant 1\\
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{10} &> \frac{9}{10}\\
\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{100} &> \frac{90}{100} = \frac{9}{10}\\
\frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{1000} &> \frac{900}{1000} = \frac{9}{10}\\
\frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{10000} &> \frac{9000}{10000} = \frac{9}{10}\\
\cdots\quad &\ \cdots
\end{aligned}$$

and so by adding both sides we see that $\sum_{n=1}^{\infty} \frac{1}{n}$ is at least as big as $1 + \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \cdots = \infty$.

Theorem B. *The series* $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ *converges for all* $x > 1$.

This was also observed in Calculus II using the Integral Test for convergence:

$$\int_1^\infty \frac{1}{t^x}\,dt = \frac{1}{x-1} < \infty$$

if $x > 1$. Recall that this does not give us the actual value of $\sum_{n=1}^{\infty} n^{-x}$, which must be determined by other methods; it does however provide upper and lower bounds

$$\frac{1}{x-1} < \zeta(x) < 1 + \frac{1}{x-1}\,.$$

For example we have $\zeta(2) = \frac{\pi^2}{6}$, as shown on another course handout.

We remark that the *Riemann zeta function* $\zeta(z)$ is in fact defined for all complex numbers $z \neq 1$, and has values $\zeta(z) \in \mathbb{C}$. The series $\sum_{n=1}^{\infty} n^{-z}$ only converges for $z = x + iy$

with $x > 1$, but this expression extends uniquely to an analytic (differentiable) function $\zeta(z)$ for all $z \in \mathbb{C}$, $z \neq 1$.

We have observed the *Euler factorization*

$$\begin{aligned}\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} &= \prod_p \Big(\frac{1}{1-\frac{1}{p^x}}\Big)\\ &= \Big(\frac{1}{1-\frac{1}{2^x}}\Big)\Big(\frac{1}{1-\frac{1}{3^x}}\Big)\Big(\frac{1}{1-\frac{1}{5^x}}\Big)\Big(\frac{1}{1-\frac{1}{7^x}}\Big)\Big(\frac{1}{1-\frac{1}{11^x}}\Big) \times \cdots,\end{aligned}$$

valid for all $x > 1$, and the product is over all primes p.

Theorem C. *The series* $\sum_p \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \cdots$ *diverges.*

Proof. Recall the geometric series

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots,$$

convergent whenever $|u| < 1$. Integrating both sides with respect to u as in Calculus II yields the Taylor series

$$\ln\big(\tfrac{1}{1-u}\big) = u + \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{u^k}{k},$$

convergent whenever $|u| < 1$ (the series also converges in fact for $u = -1$, although we will not need this fact here). Now for $x > 1$, we have

$$\begin{aligned}\ln \zeta(x) &= \sum_p \ln\left(\frac{1}{1-\frac{1}{p^x}}\right)\\ &= \sum_p \sum_{k=1}^{\infty} \frac{1}{kp^{kx}}\\ &= \sum_p \frac{1}{p^x} + \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^{kx}}\end{aligned}$$

where, as usual, p ranges over all primes. Note that $\zeta(x) \to \infty$ as $x \to 1^+$, so $\ln \zeta(x) \to \infty$ also. We have split $\ln \zeta(x)$ into two sums, and we claim that the latter sum is bounded (i.e. stays 'small') as $x \to 1^+$; indeed

$$\sum_p \sum_{k=2}^{\infty} \frac{1}{kp^{kx}} \leqslant \frac{1}{2} \sum_p \sum_{k=2}^{\infty} \frac{1}{p^{kx}}$$

and now we use the sum of the geometric series

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{1}{p^{kx}} &= \frac{1}{p^{2x}}+\frac{1}{p^{3x}}+\frac{1}{p^{4x}}+\frac{1}{p^{5x}}+\cdots\\ &= \frac{1/p^{2x}}{1-\frac{1}{p^x}} = \frac{1}{p^x(p^x-1)} \leqslant \frac{2}{p^{2x}}\end{aligned}$$

to obtain

$$\sum_p\sum_{k=2}^{\infty}\frac{1}{kp^{kx}} \leqslant \sum_p\frac{1}{p^{2x}} \leqslant \sum_{r=1}^{\infty}\frac{1}{r^2} = \zeta(2) = \frac{\pi^2}{6} < \infty\,.$$

Now recall that

$$\ln\zeta(x) = \sum_p\frac{1}{p^x} + (\text{'other terms'})$$

where the 'other terms' remain less than $\frac{\pi^2}{6}$ as $x\to 1^+$. But the left side $\zeta(x)\to\infty$ as $x\to 1^+$, so $\sum_p \frac{1}{p^x}\to\infty$ as $x\to 1^+$. This proves the result. □

A slight variation of $\zeta(x)$, in which we sum only over the *odd* positive integers n, gives the function

$$\begin{aligned}L_0(x) &= \sum_{n\text{ odd}}\frac{1}{n^x}\\ &= 1+\frac{1}{3^x}+\frac{1}{5^x}+\frac{1}{7^x}+\frac{1}{11^x}+\cdots\\ &= \Big(1+\frac{1}{3^x}+\frac{1}{9^x}+\cdots\Big)\Big(1+\frac{1}{5^x}+\frac{1}{25^x}+\cdots\Big)\Big(1+\frac{1}{7^x}+\frac{1}{49^x}+\cdots\Big)\times\cdots\\ &= \prod_{p\text{ odd}}\left(\frac{1}{1-\frac{1}{p^x}}\right)\\ &= \Big(1-\frac{1}{2^x}\Big)\zeta(x)\end{aligned}$$

whenever $x>1$. (Here the product is only over the *odd* primes $p=3,5,7,11,\ldots$.) Taking the natural logarithm of both sides gives

$$\begin{aligned}\ln L_0(x) &= \sum_{p\text{ odd}}\ln\left(\frac{1}{1-\frac{1}{p^x}}\right)\\ &= \sum_{p\text{ odd}}\sum_{k=1}^{\infty}\frac{1}{kp^{kx}}\\ &= \sum_{p\text{ odd}}\frac{1}{p^x} + (\text{'small terms'})\end{aligned}$$

where 'small terms' refers to terms which *do not* tend to infinity as $x \to 1^+$ (they remain less than some small positive constant; clearly they must remain less than $\frac{\pi^2}{6}$ as before).

Note that $L_0(x) \to \infty$ as $x \to 1^+$. (One way to see this is to recall that $L_0(x) = \big(1 - \frac{1}{2^x}\big)\zeta(x)$.) It follows (and no surprise! since this was essentially the conclusion of Theorem C) that the dominant terms

$$\sum_{p \text{ odd}} \frac{1}{p^x} \to \infty \quad \text{as } x \to 1^+.$$

The function $L_0(x)$ is an example of a *Dirichlet L-function.* In order to define another such L-function we must first define

$$\chi(n) = \begin{cases} 0, & \text{if } n \equiv 0, 2 \mod 4; \\ 1, & \text{if } n \equiv 1 \mod 4; \\ -1, & \text{if } n \equiv 3 \mod 4. \end{cases}$$

Now define

$$L_1(x) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^x} = 1 - \frac{1}{3^x} + \frac{1}{5^x} - \frac{1}{7^x} + \frac{1}{9^x} - \frac{1}{11^x} + \cdots.$$

In exactly the same way as we obtained the Euler factorization of $\zeta(z)$, we have

$$\begin{aligned} L_1(x) &= \Big(1 - \frac{1}{3^x} + \frac{1}{9^x} - \frac{1}{27^x} + \cdots\Big)\Big(1 + \frac{1}{5^x} + \frac{1}{25^x} + \frac{1}{125^x} + \cdots\Big)\Big(1 - \frac{1}{7^x} + \frac{1}{49^x} - \frac{1}{343^x} + \cdots\Big) \times \cdots \\ &= \prod_p \Bigg(\frac{1}{1 - \frac{\chi(p)}{p^x}}\Bigg). \end{aligned}$$

As usual, p ranges over all primes (although $p = 2$ contributes nothing to the sum or product in this case since $\chi(2) = 0$). Once again, taking the natural logarithm of both sides yields

$$\begin{aligned} \ln L_1(x) &= \sum_{p \text{ odd}} \ln\Bigg(\frac{1}{1 - \frac{\chi(p)}{p^x}}\Bigg) \\ &= \sum_{p \text{ odd}} \sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{kx}} \\ &= \sum_{p \text{ odd}} \frac{\chi(p)}{p^x} + \big(\text{'small terms'}\big) \end{aligned}$$

where 'small terms' refers to terms which *do not* tend to infinity as $x \to 1^+$. Note that unlike $\zeta(x)$ and $L_0(x)$, the function $L_1(x)$ *does not* tend to infinity as $x \to 1^+$; indeed

$$L_1(x) \to 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots < \infty$$

as $x \to 1^+$. (The convergence of this series follows from Leibniz' Test for alternating series; its value is in fact $\frac{\pi}{4}$, as you might have also seen in Calc II.) This means that the dominant terms converge:

$$\sum_{p \text{ odd}} \frac{\chi(p)}{p^x} \text{ has a finite limit as } x \to 1^+ .$$

By adding this to the dominant terms of $L_0(x)$, we see that

$$2 \sum_{p \equiv 1 \mod 4} \frac{1}{p^x} \to \infty \quad \text{as } x \to 1^+ ;$$

and by subtracting instead of adding, we see that

$$2 \sum_{p \equiv 3 \mod 4} \frac{1}{p^x} \to \infty \quad \text{as } x \to 1^+ .$$

This proves the following Theorem of Dirichlet:

Theorem D. *Both of the sums*

$$\sum_{p \equiv 1 \mod 4} \frac{1}{p} = \frac{1}{5} + \frac{1}{13} + \frac{1}{17} + \frac{1}{29} + \cdots$$

and

$$\sum_{p \equiv 3 \mod 4} \frac{1}{p} = \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{19} + \frac{1}{23} + \cdots$$

diverge. In particular, there are infinitely many primes $p \equiv 1 \mod 4$, *and there are infinitely many primes* $p \equiv 3 \mod 4$.