University of Wyoming

Math 3700—Spring 2016

Combinatorics

Department of Mathematics

Funny Dice

A *standard die* is a six-sided cube with faces labelled 1,2,3,4,5,6. The possible total rolls for a pair of standard dice, with the corresponding probability of occurrence in each case, are summarized in the following table:

Total Roll	**Probability**
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36
Total	$36/36 = 1$

Curiously, it is possible to achieve precisely the same outcomes for the total roll, with exactly the same corresponding probabilities, with a nonstandard pair of cube dice, in which one die has faces labelled 1,2,2,3,3,4 and the other has sides labelled 1,3,4,5,6,8. (Check this!) Why is this? and are there other possible ways of labelling the faces of a pair of cube dice that achieve the same outcomes?

A pair of dice with faces labelled 0,1,2,3,4,5 and 2,3,4,5,6,7 will achieve the same outcomes; but this variant of the standard dice is achieved by simply taking one point from the first die and transferring it to the second. Let's not count this: assume that every die has lowest number 1 appearing on its faces.

To approach our problem in a systematic manner, we first observe that the possible total rolls for a pair of standard dice, and the number of ways (out of 36 possible outcomes) of rolling each possible total, is concisely expressed by the polynomial identity

$$(x+x^2+x^3+x^4+x^5+x^6)^2 = x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}.$$

The fact that the nonstandard pair of dice described above, gives the same possible outcomes for total roll, with the same number of ways (out of 36 possible combinations) of rolling each possible total, is expressed by the polynomial identity

$$\begin{aligned}&(x+2x^2+2x^3+x^4)(x+x^3+x^4+x^5+x^6+x^8)\\ &\qquad = x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}\end{aligned}$$

which is identical to the polynomial obtained for the standard pair of dice.

We explain the relationship between the polynomial identities and the total dice roll as follows: Let $f_1(x)$ be the polynomial in which the coefficient of x^k equals the number of faces labelled k in the first die, and let $f_2(x)$ be the polynomial representing the labels on the second die in a similar manner. Then the x^n term in the polynomial $f_1(x)f_2(x)$ equals the total of all terms of the form $(ax^k)(bx^{n-k}) = abx^n$ (for all possible values of k) where ax^k is a term of degree k in $f_1(x)$, and bx^{n-k} is a term in $f_2(x)$ of degree $n-k$. But ab represents the number of ways of rolling k on the first die *and* $n-k$ on the second die; and by summing over all values of k we take account of all possible ways of obtaining a total roll of n.

The possible outcomes for the total roll, each with probability designated by the table above, are neatly expressed by the polynomial

$$f(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

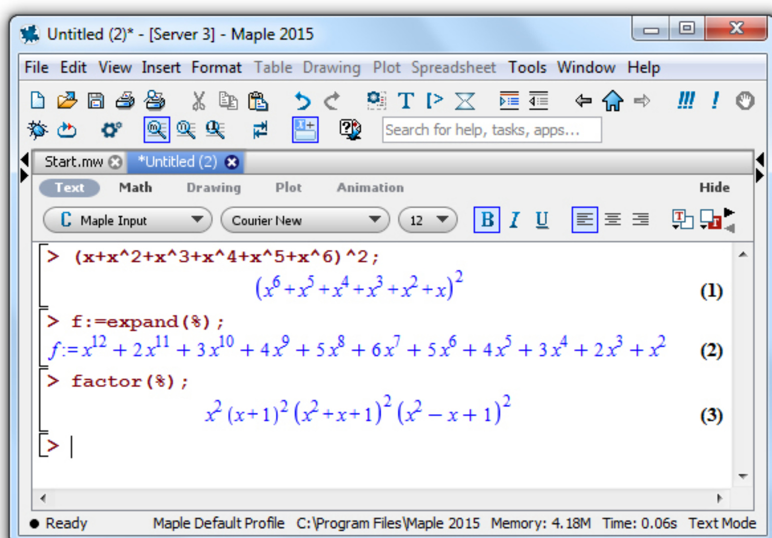
The problem at hand amounts to finding suitable factorizations of the form

$$f(x) = f_1(x)f_2(x)$$

where the polynomial factors $f_1(x)$ and $f_2(x)$ represent the labels on the faces of the two dice. We solve this problem in very much the same way as one might determine all possible factorizations of an integer; for example the only ways to factor 12 as a product of two positive integers are given by $12 = 3 \times 4 = 2 \times 6 = 1 \times 12$. (We do not count the factorization 4×3 as distinct from 3×4.) To see that these are the only possible ways to factor $12 = ab$, it suffices to consider the prime factorization $12 = 2 \times 2 \times 3$ and then to consider all possible ways of grouping the prime factors together into the two factors a and b. To use a similar strategy for the polynomial $f(x)$, we first find the irreducible factors of $f(x)$:

$$f(x) = x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2.$$

This factorization can be found ether by hand, or by using the software MAPLE available over the UW network. Here is a typical session in MAPLE:



Note that the symbol '%' in MAPLE is an abbreviation for 'the preceding expression'.

Two possible pairs of polynomial factors are given by

$$f_1(x) = f_2(x) = x(x+1)(x^2+x+1)(x^2-x+1) = x + x^2 + x^3 + x^4 + x^5 + x^6$$

or

$$f_1(x) = x(x+1)(x^2+x+1) = x + 2x^2 + 2x^3 + x^4;$$
$$f_2(x) = x(x+1)(x^2+x+1)(x^2-x+1)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8.$$

These are the two solutions identified previously. One might consider a possibility such as

$$f_1(x) = x(x+1)(x^2+x+1)^2 = x + 3x^2 + 5x^3 + 5x^4 + 3x^5 + x^6;$$
$$f_2(x) = x(x+1)(x^2-x+1)^2 = x - x^2 + x^3 + x^4 - x^5 + x^6;$$

but these factors are not suitable since $f_2(x)$ has some negative coefficients (this polynomial would require that the second die has -1 faces labelled 2, and -1 faces labelled 5) and because the coefficients of $f_1(x)$ add up to 18 rather than 6 (if the first die has one side labelled 1, three sides labelled 2, ..., one side labelled 6, then this is a total of eighteen sides!). So in order for the polynomial factors $f_1(x)$ and $f_2(x)$ to be suitable, their coefficients must be non-negative integers adding up to 6. The factorization

$$f_1(x) = x(x^2+x+1)^2 = x + 2x^2 + 3x^3 + 2x^4 + x^5;$$
$$f_2(x) = x(x+1)^2(x^2-x+1)^2 = x - x^2 + x^3 + x^4 - x^5 + x^6$$

is unsuitable for similar reasons. Moreover the factors

$$f_1(x) = x^2(x+1)(x^2+x+1) = x^2 + 2x^3 + 2x^4 + x^5;$$
$$f_2(x) = (x+1)(x^2+x+1)(x^2-x+1) = 1 + x^2 + x^3 + x^4 + x^5 + x^7,$$

although suitable, represent only a slight modification of the non-standard dice listed earlier: simply subtract 1 from the faces of one die, and add 1 to the faces of the other die. We don't bother to list this as a separate solution.

Following the guidelines listed above, we find that there are essentially only two ways to label the sides of a pair of cube dice, to obtain the table of probabilities given above: either the standard labelling, or the non-standard labelling 1,2,2,3,3,4 and 1,3,4,5,6,8. We do not list explicitly all possible ways to partition the irreducible factors of $f(x)$ between the two polynomial factors $f_1(x)$ and $f_2(x)$ since this is a routine matter, as is the process of checking the various factorizations that arise to see which are suitable.

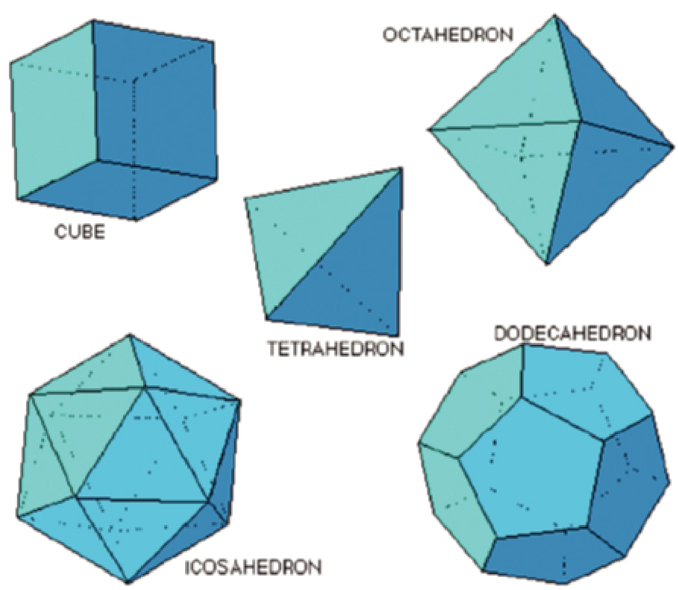
As an example of an application of polynomials, this is remarkable for several reasons. In particular we have made no mention of values of polynomial functions: at no point have we substituted numerical values for x or asked for corresponding numerical values for $f(x)$ or $f_1(x)$ or $f_2(x)$. The symbol x is merely a placeholder, with no numerical value! Actually each of the polynomials $f(x)$, $f_1(x)$, $f_2(x)$ is an instance of a generating function for a sequence; for example the sequence $1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1$ (the number of ways of rolling $n \in \{0, 1, 2, \ldots\}$ with two standard dice) has $f(x)$ as its generating function;

since only finitely many values are possible for the roll, the series simply reduces to a polynomial.

HOMEWORK #4 Due Friday, April 8

1. Consider a pair of tetrahedral dice, each die having four triangular faces with labels 1,2,3,4. What are the possibilities for an alternative pair of tetrahedral dice with different face labels, but having the same possible outcomes for the total roll, with the same corresponding probabilities?
2. Repeat Question 1 for a pair of dodecahedral dice, each die having twelve pentagonal faces with labels 1,2,3,…,12.
3. Repeat Question 1 for a pair of octahedral dice, each die having eight triangular faces with labels 1,2,3,…,8.

The Five Platonic Solids



This Maple code finds all pairs of n-sided dice giving the same probability distribution of total value of the roll, as the standard pair. It uses the method of generating functions discussed in class. *For brevity I have omitted the extraneous factors of x in each of the polynomials.*

We first write a subroutine to test whether a polynomial $f(x)$ is feasibly the generating function for the labels of an n-sided die. For this we must check that its coefficients are non-negative and that they add up to 1:

```
> with(PolynomialTools):
> feasible:=proc(f,n)
>    local coeff,sumcoeffs: sumcoeffs:=0:
>    for coeff in CoefficientList(f,x) do
>       if coeff<0 then return false; fi:
>       sumcoeffs:=sumcoeffs+coeff: od:
>    return evalb(sumcoeffs=n); end:
```

The following subroutine takes as input the generating function $f(x)$ for labels of an n-sided die, and returns the list of labels on its n faces:

```
> face_labels:=proc(f)
>    local list_labels,i,j,c:
>    list_labels:=[]: i:=1:
>    for c in CoefficientList(f,x) do
>       for j from 1 to c do
>         list_labels:=[op(list_labels),i]:
>         od: i:=i+1: od:
>    return list_labels; end:
```

The main routine:

```
> find_dice:=proc(n)
>    local i,f,f1,f2,ffact,nfact,fbase,fexp,fexp_all,sol,sols:
>    f:=simplify((x^n-1)/(x-1));
>    ffact:=factors(f)[2]: nfact:=nops(ffact):
>    fbase:=[seq(ffact[i][1],i=1..nfact)]:
>    fexp_all:=cartprod([seq([0,1,2],i=1..nfact)]):
>    sols:={}:
>    while not fexp_all[finished] do
>       fexp:=fexp_all[nextvalue]():
>       f1:=product(fbase[i]^fexp[i],i=1..nfact):
>       f2:=factor(simplify(f^2/f1)):
>       if feasible(f1,n) and feasible(f2,n) then
>          sols:=sols union {sort([sort(f1),sort(f2)])}: fi: od:
>    for sol in sols do
>       print(sol,face_labels(sol[1]),face_labels(sol[2])); od:
>    end:
```

As a test case, we verify that there are exactly two pairs of 6-sided dice with the same probability distribution as the standard pair. One of these is the standard pair itself, and the other one is the nonstandard pair:

```
> find_dice(6);
```

$$\left[(x^2+x+1)\,(x+1),\ (x^2-x+1)^2\,(x^2+x+1)\,(x+1)\right],\ [1,2,2,3,3,4],\ [1,3,4,5,6,8]$$
$$\left[(x+1)\,(x^2+x+1)\,(x^2-x+1),\ (x+1)\,(x^2+x+1)\,(x^2-x+1)\right],\ [1,2,3,4,5,6],\ [1,2,3,4,5,6] \qquad \textbf{(1)}$$

Next we find two pairs of tetrahedral dice, including the standard pair:

```
> find_dice(4);
```

$$\left[(x+1)^2, (x^2+1)^2\right], [1, 2, 2, 3], [1, 3, 3, 5]$$
$$\left[(x+1)(x^2+1), (x+1)(x^2+1)\right], [1, 2, 3, 4], [1, 2, 3, 4] \tag{2}$$

For $n = 8$ there are four pairs of octrahedral dice, including the standard pair:

```
> find_dice(8);
```

$$\left[(x+1)(x^4+1)^2, (x^2+1)^2(x+1)\right], [1, 2, 5, 5, 6, 6, 9, 10], [1, 2, 3, 3, 4, 4, 5, 6]$$
$$\left[(x+1)^2(x^2+1), (x^2+1)(x^4+1)^2\right], [1, 2, 2, 3, 3, 4, 4, 5], [1, 3, 5, 5, 7, 7, 9, 11]$$
$$\left[(x+1)^2(x^4+1), (x^2+1)^2(x^4+1)\right], [1, 2, 2, 3, 5, 6, 6, 7], [1, 3, 3, 5, 5, 7, 7, 9]$$
$$\left[(x+1)(x^2+1)(x^4+1), (x+1)(x^2+1)(x^4+1)\right], [1, 2, 3, 4, 5, 6, 7, 8], [1, 2, 3, 4, 5, 6, 7, 8] \tag{3}$$

For $n = 12$ there are eight pairs of dodecahedral dice, including the standard pair:

```
> find_dice(12);
```

$$\left[(x^2+1)(x^2+x+1)(x+1), (x+1)(x^2-x+1)^2(x^4-x^2+1)^2(x^2+1)(x^2+x+1)\right], [1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 6], [1, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 18]$$
$$\left[(x^2+1)^2(x^4-x^2+1)(x^2+x+1), (x+1)^2(x^2+x+1)(x^2-x+1)^2(x^4-x^2+1)\right], [1, 2, 3, 3, 4, 5, 7, 8, 9, 9, 10, 11], [1, 2, 4, 5, 5, 6, 8, 9, 9, 10, 12, 13]$$
$$\left[(x^2+1)^2(x^4-x^2+1)^2(x^2+x+1), (x^2-x+1)^2(x^2+x+1)(x+1)^2\right], [1, 2, 3, 7, 7, 8, 8, 9, 9, 13, 14, 15], [1, 2, 3, 4, 4, 5, 5, 6, 6, 7, 8, 9]$$
$$\left[(x^2-x+1)(x^2+x+1)(x+1)^2, (x^2+1)^2(x^4-x^2+1)^2(x^2+x+1)(x^2-x+1)\right], [1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7], [1, 3, 5, 7, 7, 9, 9, 11, 11, 13, 15, 17]$$
$$\left[(x+1)(x^2+x+1)(x^2-x+1)(x^2+1), (x^2+1)(x^4-x^2+1)^2(x^2+x+1)(x+1)(x^2-x+1)\right], [1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 8], [1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16]$$
$$\left[(x^2+1)(x^4-x^2+1)(x^2+x+1)(x+1), (x+1)(x^2+1)(x^2-x+1)^2(x^4-x^2+1)(x^2+x+1)\right], [1, 2, 2, 3, 3, 4, 7, 8, 8, 9, 9, 10], [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14]$$
$$\left[(x^2+1)^2(x^2-x+1)(x^4-x^2+1)(x^2+x+1), (x^2-x+1)(x^4-x^2+1)(x^2+x+1)(x+1)^2\right], [1, 3, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13], [1, 2, 2, 3, 5, 6, 6, 7, 9, 10, 10, 11]$$
$$\left[(x+1)(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1), (x+1)(x^2+1)(x^2+x+1)(x^2-x+1)(x^4-x^2+1)\right], [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] \tag{4}$$