Using Continued Fractions to Recognize Rationals and Quadratic Irrationals

In HW6 you are given decimal expansions for a rational number α and a quadratic irrational number β, and asked to find exact expressions for these numbers. We demonstrated how this is done by working through some examples in class; and here are two more examples.

Example 1. Identify the rational number α with decimal approximation $\alpha \approx 1.57352941176$.

Solution.

$$\alpha \approx 1+\frac{1}{1.74358974359} \approx 1+\frac{1}{1+\frac{1}{1.34482758621}} \approx 1+\frac{1}{1+\frac{1}{1+\frac{1}{2.9}}}$$

$$\approx 1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1.111111111111}}}} \approx 1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{9}}}}} = [1,1,1,2,1,9]\,.$$

Thus

$$\alpha = 1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{9}}}}} = 1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{9}{10}}}} = 1+\frac{1}{1+\frac{1}{1+\frac{10}{29}}}$$

$$= 1+\frac{1}{1+\frac{29}{39}} = 1+\frac{39}{68} = \frac{107}{68}\,.$$

Check: Using a calculator, $\frac{107}{68} \approx 1.57352941176$.

Note that the actual numerical values will depend on the limited precision of your calculator. The more steps required to find a pattern in the continued fraction expansion of α, the greater the accumulation of roundoff error. For example a typical model of handheld calculator may obtain 8.999999973 instead of 9 in the last step. At this point you would exercise your own judgment in recognizing 9 as the correct value, and attributing the discrepancy to roundoff error.

Example 2. Identify the quadratic irrational number β with decimal approximation $\beta \approx 1.74641016151$.

Solution. As in Example 1, we determine the continued fraction expansion of β as $\beta = [1,1,2,1,16,1,1,1,16,1,1,\ldots] = [1,1,2,\overline{1,16,1,1}\,]$. Let $\gamma = [\,\overline{1,16,1,1}\,]$, so that

$$\gamma \;=\; 1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}} \;=\; 1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\gamma}}}}\;.$$

Then

$$\gamma = 1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{\gamma}{\gamma+1}}} \;=\; 1+\cfrac{1}{16+\cfrac{2\gamma+1}{\gamma+1}} \;=\; 1+\frac{2\gamma+1}{33\gamma+17} \;=\; \frac{35\gamma+18}{33\gamma+17}$$

and so we must solve the quadratic equation $33\gamma^2 - 18\gamma - 18 = 0$ for γ:

$$\gamma = \frac{18 \pm \sqrt{18^2 + 4{\cdot}18{\cdot}33}}{66} = \frac{3 \pm 5\sqrt{3}}{11}\;.$$

Since $\gamma > 0$, we must have $\gamma = (3+5\sqrt{3})/11$ and

$$\beta \;=\; 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\gamma}}} \;=\; 1+\cfrac{1}{1+\cfrac{\gamma}{2\gamma+1}} \;=\; 1+\frac{2\gamma+1}{3\gamma+1} \;=\; \frac{5\gamma+2}{3\gamma+1} \;=\; \frac{5\left(\frac{3+5\sqrt{3}}{11}\right)+2}{3\left(\frac{3+5\sqrt{3}}{11}\right)+1}$$

$$=\; \frac{5(3+5\sqrt{3}\,)+22}{3(3+5\sqrt{3}\,)+11} \;=\; \frac{1}{5}\left(\frac{37+25\sqrt{3}}{4+3\sqrt{3}}\right)\left(\frac{4-3\sqrt{3}}{4-3\sqrt{3}}\right) \;=\; \frac{1}{5}\left(\frac{-77-11\sqrt{3}}{-11}\right) \;=\; \frac{7+\sqrt{3}}{5}\;.$$

Check: Using a calculator, $(7+\sqrt{3})/5 \approx 1.74641016151$.