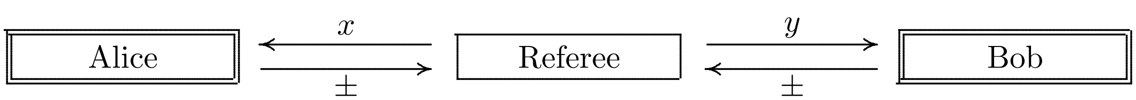
Bell's Theorem

The CHSH Game

We motivate this topic by considering the CHSH game, in which two participants, Alice and Bob, are separated by a vast distance. (This is a cooperative game, and we must exclude the possibility that they are able to cheat by communicating.) Every round of the game requires that they each send a single bit '+' ('up') or '−' ('down') to a referee located at their midpoint. Assuming they had an opportunity to collude beforehand, it is clear that they can conspire to make their answers agree every time; for example, they can agree to both always say 'up', or both always say 'down'; or they can both respond according to some pre-agreed sequence sequence such as $--+---++\cdots$ (determined by the parity of the digits $3.1415926\ldots$). If instead they are required to *disagree* on every round, it would again be easy for Alice and Bob to win on every round (e.g. Alice always says '+' and Bob always says '−').

The actual CHSH game, however, is less trivial due to the use of hidden randomness, which makes the optimal stragegy less clear. Every round, the referee flips two coins, yielding a pair of random bits $x, y \in \{0, 1\}$. The referee sends x to Alice, and y to Bob, both of whom must respond with '+' or '−'. When $(x, y) \in \{(0, 0), (0, 1), (1, 0)\}$, Alice and Bob must try to agree in their responses; but when $(x, y) = (1, 1)$, Alice and Bob must try to disagree in their responses. But since Alice cannot know y, and Bob cannot know x, they do not even know (half the time) whether they are trying to agree or disagree.



A strategy that allows Alice and Bob to win 75% of the time, is for both of them to simply respond '+' every time, regardless of the outcomes of coin flips. Classical reasoning leads to the conclusion that this strategy is optimal. The startling prediction of quantum mechanics, however, is that it is possible to do better if Alice and Bob are able to make use of quantum entanglement. If they are able to generate n EPR pairs of particles beforehand, with Alice and Bob both taking one particle from each pair to their distant location, they can in fact win a little over 85% of the time. In his 1964 theorem, John Bell proved an inequality showing that 75% is the maximum success rate for Alice and Bob under most natural hidden variable theories. Bell's Theorem is therefore seen as a 'no-go' theorem for hidden variable theories (not excluding altogether the possibility of hidden variables, but making it very hard to imagine a reasonable set of assumptions that would support belief in hidden variables). The predictions of quantum mechanics, notably Bell's work, have been verified in the lab many times since then, using various versions of the CHSH game (attaining something like 84% success rate for Alice and Bob). A few decades after John Bell's death in 1990, the 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser and Anton Zeilinger for their laboratory verification of Bell's work.

John Bell   Alain Aspect   John Clauser   Anton Zeilinger

Spin Measurement of Single Electrons

Recall the ket notation for vectors $|\psi\rangle = \left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] \in \mathbb{C}^2$. The bra notation $\langle\phi| = [a\ b] : \mathbb{C}^2 \to \mathbb{C}$ denotes the linear functional mapping $|\psi\rangle \mapsto \langle\phi|\psi\rangle = [a\ b]\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] = a\alpha+b\beta$. We also denote $\langle\psi| = |\psi\rangle^* = [\overline{\alpha}\ \overline{\beta}]$ where $A^* = \overline{A^T}$ is the Hermitian conjugate (i.e. conjugate transpose) of an arbitrary complex matrix A, satisfying $(AB)^* = B^*A^*$.

Denote by S^3 the set of unit vectors in $\mathbb{C}^2$. These represent spin states of electrons (our favourite examples of qubits, i.e. quantum bits). Thus S^3 is the set of all $|\psi\rangle \in \mathbb{C}^2$ satisfying

$$\big||\psi\rangle\big|^2 = \langle\psi|\psi\rangle = [\overline{\alpha}\ \overline{\beta}]\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right] = |\alpha|^2 + |\beta|^2 = 1.$$

In order for Alice to measure the spin of an electron in spin state $|\psi\rangle \in S^3$, she first chooses an orthonormal basis of $\mathbb{C}^2$, i.e. a pair of vectors $|+_A\rangle, |-_A\rangle \in S^3$ satisfying

$$\langle +_A|+_A\rangle = \langle -_A|-_A\rangle = 1, \qquad \langle +_A|-_A\rangle = \langle -_A|+_A\rangle = 0.$$

(This amounts to choosing an axis or direction in $\mathbb{R}^3$ with respect to which she measures the spin. This connection is explained later as we don't need it now.) The two states $|+_A\rangle$ and $|-_A\rangle$ are Alice's designated 'up' and 'down' spin states, respectively. The spin measurement yields a single classical bit of information, 'up' or 'down'. These outcomes are observed with probabilities $|\langle +_A|\psi\rangle|^2$ and $|\langle -_A|\psi\rangle|^2$ respectively, according to Born's Rule. Immediately upon measurement, the spin state of her electron collapses into the corresponding state in which it was observed, i.e. $|+_A\rangle$ or $|-_A\rangle$; and subsequent measurements (with respect to the same basis) will yield the same outcome as before. A little algebra shows that $|\langle +_A|\psi\rangle|^2 + |\langle -_A|\psi\rangle|^2 = 1$, as required for probabilities of complementary events.

Spin Measurement of EPR Pairs

Consider an EPR pair of electrons e1 and e2 in the joint spin state

$$\tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \tfrac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\otimes|1\rangle.$$

Here, I denote the standard basis of $\mathbb{C}^2$ by $|0\rangle = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $|1\rangle = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. These are the same as the eigenstates $|+_z\rangle, |-_z\rangle$ for measurement in the vertical direction. If one of the two electrons is measured in the vertical direction, it has 50% chance of being detected in a spin 'up' state; and a 50% chance of being found in the spin 'down' state; but at that instant it is known that the other electron will be found to be in that same spin state if measured.

Alice has taken e1, and Bob has taken e2, to far distant locations. They have taken care not to disturb their electrons, as any interaction with the environment will result in loss of entanglement. In order for Alice and Bob to utilize such an EPR pair in their strategy for the CHSH game, our first concern is: If Alice and Bob measure their respective electrons e1 and e2 separately, using different bases which they chosen independently, what is the probabity that they obtain the same result ('up' or 'down') for their spin measurements? We will show that this probability is simply $|\langle +_A|+_B\rangle|^2$.

Before proceeding with our proof of this simple and satisfying formula, observe that $|\langle +_A|+_B\rangle|^2 \in [0,1]$ as required for a probability. Note the simplicity of this formula, which (surprisingly or not) is independent of the original choice of state of the electron pair e1, e2 (as long as e1 and e2 are maximally entangled). Especially we note that it is symmetric since $|\langle +_A|+_B\rangle| = |\overline{\langle +_B|+_A\rangle}| = |\langle +_A|+_B\rangle|$. Why should this symmetry hold? If Alice measures before Bob, then e1 (originally in the state $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$) collapses instantly into her observed eigenstate (either $|+_A\rangle$ or $|-_A\rangle$, each with certain probabilities); and then by virtue of entanglement, Bob's electron e2 is in that same state ($|+_A\rangle$ or $|-_A\rangle$). Bob's measurement will then put e2 into one of his base states (either $|+_A\rangle$ or $|-_A\rangle$, again with certain probabilities), but Alice's electron e1 (which is now disentangled from Bob's electron e2) will remain in state $|+_A\rangle$ or $|-_A\rangle$. The precise sequence of events depends on who measured first, but the final outcome, including the probability of Alice and Bob agreeing, is the same. This symmetry is expected because, as we know from special relativity, different inertial observers will not always disagree on who (Alice or Bob) did their measurement first. However, there is no way for other inertial observers to test whether Alice 'altered' e2 or Bob 'altered' e1, as those electrons have been shielded from the environment. We do not find any contradiction here between the predictions of quantum mechanics and the predictions of special relativity.

So let us proceed to prove the probability formula, assuming Alice measures first. Alice uses her preferred orthonormal basis $|+_A\rangle, |-_A\rangle$, and Bob uses his basis $|+_B\rangle, |-_B\rangle$ chosen independently. Consider the possibilities for Alice's measurement.

- Alice holds e1, initially in state $|\psi_1\rangle$. She measures it using her basis $|+_A\rangle, |-_A\rangle$, and finds e1 to be in state $|+_A\rangle$ with probability $|\langle +_A|\psi_1\rangle|^2$. In this case, Bob's electron e2 will also be in state $|+_A\rangle$, by virtue of quantum entanglement. The conditional probability that Bob also finds e2 to be in his spin 'up' state $|+_B\rangle$ (given Alice's measurement), is $|\langle +_B|+_A\rangle|^2$. So the probability that both Alice and Bob find their electron to be spin 'up' is $|\langle +_A|\psi_1\rangle|^2|\langle +_B|+_A\rangle|^2$.
- Alice measures e1 and finds it to be in state $|-_A\rangle$ with probability $|\langle -_A|\psi_1\rangle|^2$. In this case, Bob's electron e2 will also be in state $|-_A\rangle$, by virtue of quantum entanglement. The conditional probability that Bob also finds e2 to be in his spin 'down' state $|-_B\rangle$ (given Alice's measurement), is $|\langle -_B|-_A\rangle|^2$. So the probability that both Alice and Bob find their electron to be spin 'down' is $|\langle -_A|\psi_1\rangle|^2|\langle -_B|-_A\rangle|^2$.

So the total probability that Alice and Bob agree on the spin direction for their individual measurements (i.e. both up or both down) is

$$\Pr(\text{Alice and Bob agree}) = |\langle +_A|\psi_1\rangle|^2|\langle +_B|+_A\rangle|^2 + |\langle -_A|\psi_1\rangle|^2|\langle -_B|-_A\rangle|^2.$$

However,

$$|\langle +_B|+_A\rangle|^2 \;=\; 1 - |\langle -_B|+_A\rangle|^2 \;=\; |\langle -_B|-_A\rangle|^2 \;=\; |\langle +_A|+_B\rangle|^2,$$

so

$$\begin{aligned}\Pr(\text{Alice and Bob agree}) &= \Big(|\langle +_A|\psi_1\rangle|^2| + |\langle -_A|\psi_1\rangle|^2\Big)|\langle +_A|+_B\rangle|^2\\ &= |\langle +_A|+_B\rangle|^2\end{aligned}$$

as claimed.

Finally, observe that if Alice and Bob choose identical bases, they will always agree on the observed spin (both say 'up' or both say 'down'). This is clear from our description of the experiment. Moreover, our formula gives the probability of agreement as $|\langle +_A|+_A\rangle|^2 = 1$ in this case. Or if Bob's basis is the reversal of Alice's basis (i.e. $|+_B\rangle = |-_A\rangle$ and $|-_B\rangle = |+_A\rangle$) then clearly they will disagree on the spin measurement of their electrons. This is predicted by our formula, which gives the probability of agreement as $|\langle +_A|-_A\rangle|^2 = 0$.

Using Entanglement to Succeed at the CHSH Game

Alice has two choices of basis:

- If Alice learns $x = 0$ from the referee, she uses the basis
$$|+_{A_0}\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad |-_{A_0}\rangle = \begin{bmatrix}0\\1\end{bmatrix}.$$
- If Alice learns $x = 1$ from the referee, she uses the basis
$$|+_{A_1}\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \qquad |-_{A_1}\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$$

Similarly, Bob has two choices of basis:

- If Bob learns $y = 0$ from the referee, he uses the basis
$$|+_{B_0}\rangle = \tfrac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}-1\\1+\sqrt{2}\end{bmatrix}, \qquad |-_{B_0}\rangle = \tfrac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\1\end{bmatrix}.$$

- If Bob learns $y = 1$ from the referee, he uses the basis
$$|{+_{B_1}}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1\\1+\sqrt{2}\end{bmatrix}, \qquad |{-_{B_1}}\rangle = \frac{1}{\sqrt{4+2\sqrt{2}}}\begin{bmatrix}1+\sqrt{2}\\-1\end{bmatrix}.$$

At the start of each round of the CHSH game, Alice and Bob load up their matched electrons in their Stern-Gerlach devices, oriented to measure with respect to the bases A_x and B_y respectively, as designed above. Each of them detects the spin of their electron to be '+' or '−' with respect to the selected basis. Each responds with '+' or '−' to the referee, exactly as found by their measuring device. The probability that they agree is easily calculated to be

$$|\langle +_{A_x}|+_{B_y}\rangle|^2 = \begin{cases} \frac{2+\sqrt{2}}{4}, & \text{if } (x,y) \in \{(0,0),(0,1),(1,0)\}; \\ 1 - \frac{2+\sqrt{2}}{4}, & \text{if } (x,y) = (1,1). \end{cases}$$

According to the rules of the CHSH game, this means their probability of winning is $\frac{2+\sqrt{2}}{4} \approx 85.4\%$ on each round.

Further work of Tsirelson shows that this success rate is the best that can be achieved by any such experiment using the limitations of quantum mechanics as we know it. And some have suggested that we could do even better at the CHSH game if the universe turns out to be even stranger than we think (as some alternative theories suggest)... but that appears to be pure speculation.

Pauli Spin Matrices and Axes of Spin Measurement

Recall the three basic Pauli spin operators

$$\sigma_x = \begin{bmatrix}0&1\\1&0\end{bmatrix}, \qquad \sigma_y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}, \qquad \sigma_z = \begin{bmatrix}1&0\\0&-1\end{bmatrix},$$

each of which gives an isometry $S^3 \to S^3$ (i.e. unitary transformation). Their eigenstates (spin 'up' and 'down' states) are

$$\begin{aligned} |+_x\rangle &= \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad |-_x\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}; \\ |+_y\rangle &= \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}, \quad |-_y\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}; \\ |+_z\rangle &= \begin{bmatrix}1\\0\end{bmatrix}, \quad |-_z\rangle = \begin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

respectively. Thus $\sigma_x|+_x\rangle = |+_x\rangle$ and $\sigma_x|-_x\rangle = -|-_x\rangle$; and similarly for the other two operators. Given an arbitrary unit vector $\mathbf{n} = (n_x, n_y, n_z)$ (so $n_x, n_y, n_z \in \mathbb{R}$ with $n_x^2 + n_y^2 + n_z^2 = 1$), there is a corresponding spin operator

$$\mathbf{n} \cdot \boldsymbol{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix}$$

where we have used the notational convenience $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Note that each of the spin matrices is a traceless Hermitian matrix. ('Traceless' means its trace is zero, where the trace is the sum of the diagonal entries. 'Hermitian', or 'self-adjoint', means that $\sigma^* = \sigma$.)

Each spin matrix is diagonalizable with eigenvalues $+1$ and -1; and it has a corresponding orthonormal basis of eigenvectors, called *eigenstates*.

The four bases of type A_0, A_1, B_0, B_1 which played such a crucial role in the Bell experiment described above, are the bases of eigenstates of the Pauli operators $\mathbf{n} \cdot \boldsymbol{\sigma}$ as $\mathbf{n}$ ranges over the four unit vectors

$$(0,0,1), \quad (1,0,0), \quad \tfrac{1}{\sqrt{2}}(1,0,1), \quad \tfrac{1}{\sqrt{2}}(1,0,-1) \in \mathbb{R}^3.$$

These four choices of $\mathbf{n}$ are used to specify the physical orientation of the axes in $\mathbb{R}^3$ when using a Stern-Gerlach apparatus to measure the electron spin. These vectors all lie in the xz-plane, as a choice of convenience so that the resulting Pauli spin matrices have real entries, but this feature is not essential. Note that the angles between these four physical axes are spaced at angles of 45°, whereas the corresponding eigenvectors are spaced at angles of 22.5°; e.g. $|-_{B_0}\rangle = \left[\begin{smallmatrix} \cos 22.5^\circ \\ \sin 22.5^\circ \end{smallmatrix}\right]$. This echoes our discussion in class about spin vectors move around a circle at half their rate as their physical counterparts.