The Basel Problem

Evaluation of $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$ was a problem that eluded the Bernoullis. The identity $\zeta(2) = \frac{\pi^2}{6}$ is generally attributed to Euler, who supplied an heuristic proof for this identity (given in my handout 'Needles and Numbers' (`http://ericmoorhouse.org/handouts/needles_and_numbers.pdf`)). Basel, Switzerland, home to Euler and the Bernoullis, gives its name to this problem. You will note that Euler's argument, while elementary, has a gap (later filled by the Weierstrass Factorization Theorem). An alternative to Euler's elegant approach uses Fourier analysis. But while many different proofs of Euler's identity are now known, many are still interested in the challenge of finding the most elementary proof available.

Recently, the 'lighthouse' proof has been offered as the most elementary proof yet, and the one which most compellingly finds a connection between $\zeta(2)$ and geometric circles; see `https://www.youtube.com/watch?v=d-o3eB9sfls`. Their proof provides an elementary argument; yet at the crucial step, a careful argument is required to express the limit correctly. The video omits the rigorous argument at this step (which I will provide below), relying on graphics to convince the viewer of the validity of the necessary limit. Note that at certain key steps of the argument, the video replaces algebraic identities with synthetic arguments from plane geometry. (Such arguments rely on statements about congruence of angles and line segments, in place of algebraic identities.) I will replace these synthetic constructions with algebraic formulas; so without sacrificing the elementary nature of the argument, I will not need to cite basic theorems from synthetic plane geometry. (Of course in doing so, I will sacrifice the charm of using arguments from synthetic plane geometry.)

For each positive integer n, we denote $\zeta_n = e^{2\pi i/n}$, a complex primitive n-th root of unity. (This notation is standard; and hopefully it will not be confused with our notation for the Riemann zeta function $\zeta(s)$, which is also standard. Note that $\zeta(s)$ will not be shown with any subscripts.)

We challenge the reader to verify that for every positive integer N,

$$\sum_{k=1}^{N} \frac{1}{|1-\zeta_{4N}^{2k-1}|^2} = \frac{N^2}{2}.$$

Note that the values ζ_{4N}^{2k-1} are selected vertices of a regular $4N$-gon inscribed in the unit circle $|z| = 1$ in $\mathbb{C}$; but we only use one-quarter of the vertices, these being alternate vertices on the semicircle lying in the upper half-plane. This observation is the key linking our exposition to the version appearing in the video. In fact we do not require the identity for all positive integers N; it suffices to consider an unbounded sequence of N-values. And so, just as in the video, we consider only the powers of two, namely $N = 2^{n-1}$; and we prove the identity only in this special case:

Theorem. For every positive integer n, we have $\displaystyle\sum_{k=1}^{2^{n-1}} \frac{1}{|1-\zeta_{2^{n+1}}^{2k-1}|^2} = 2^{2n-3}$.

Proof. We denote $w_n = \sum_{k=1}^{N} \frac{1}{|1-\zeta_{4N}^{2k-1}|^2}$ where $N = 2^{n-1}$, $n \geqslant 1$. For $n = 1$, we have $\zeta_4 = i$ and the sum has a single term $w_1 = \frac{1}{|1-i|^2} = \frac{1}{2}$. The identity holds in this case. Now observe that $w_n = \frac{1}{2}\sum_{k=1}^{2N} \frac{1}{|1-\zeta_{4N}^{2k-1}|^2}$ and evaluate by induction:

$$\begin{aligned} w_{n+1} &= \tfrac{1}{2}\sum_{k=1}^{2N} \frac{1}{|1-\zeta_{8N}^{2k-1}|^2} = \tfrac{1}{2}\sum_{k=1}^{N} \frac{1}{|1-\zeta_{8N}^{2k-1}|^2} + \tfrac{1}{2}\sum_{k=N+1}^{2N} \frac{1}{|1-\zeta_{8N}^{2k-1}|^2} \\ &= \tfrac{1}{2}\sum_{k=1}^{N}\left(\frac{1}{|1-\zeta_{8N}^{2k-1}|^2} + \frac{1}{|1-\zeta_{8N}^{2N+2k-1}|^2}\right) = \tfrac{1}{2}\sum_{k=1}^{N}\left(\frac{1}{|1-\zeta_{8N}^{2k-1}|^2} + \frac{1}{|1+\zeta_{8N}^{2k-1}|^2}\right). \end{aligned}$$

Now using $|z|^2 = z\overline{z}$ we obtain the identity $\frac{1}{|1-z|^2} + \frac{1}{|1+z|^2} = \frac{2(1+|z|^2)}{|1-z^2|^2}$, from which we easily obtain $w_{n+1} = 4w_n$. Using the initial value $w_1 = \frac{1}{2}$ together with this recurrence relation gives $w_n = 2^{2n-3}$ as required. □

Noting that

$$\begin{aligned} |1-\zeta_{4N}^{2k-1}|^2 &= (1-\zeta_{4N}^{2k-1})(1-\overline{\zeta_{4N}^{2k-1}}) = 2-\zeta_{4N}^{2k-1}-\overline{\zeta_{4N}^{2k-1}} \\ &= 2\big(1-\cos\tfrac{(2k-1)\pi}{2N}\big) = 4\sin^2\tfrac{(2k-1)\pi}{4N}, \end{aligned}$$

we obtain

Corollary. For every positive integer n, denoting $N = N(n) = 2^{n-1}$, we have

$$\sum_{k=1}^{N} \frac{1}{N^2\sin^2\frac{(2k-1)\pi}{4N}} = 2.$$

Now as $n \to \infty$, $N \to \infty$ and for each fixed k,

$$\frac{1}{N^2\sin^2\frac{(2k-1)\pi}{4N}} = \frac{16}{(2k-1)^2\pi^2} + O(N^{-2}).$$

Meanwhile the number of terms also tends to infinity, so the Corollary apparently gives

$$\sum_{k=1}^{\infty} \frac{16}{(2k-1)^2\pi^2} = 2, \quad \text{i.e.} \quad 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

We are almost there! Recall the Euler factorization

$$\zeta(2) = \prod_p \frac{1}{1-\frac{1}{p^2}} = \frac{1}{1-\frac{1}{2^2}} \prod_{p>2} \frac{1}{1-\frac{1}{p^2}} = \frac{4}{3}\Big(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots\Big).$$

Combining this with the previous identity yields $\zeta(2) = \frac{\pi^2}{6}$ as required.

But not so fast! The alert student will observe that some care is needed in justifying our value for the limit as $N \to \infty$ since both the number of terms, *and* the values of the individual terms, depends on N. In general, sloppiness can lead to nonsensical results! For example, let's evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3}$. For each fixed k, the individual terms of the sum tends to zero; then we take the sum of n zeroes, giving zero for the final limit, right? Wrong! Actually, $\int_0^1 x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)/6}{n^3} = \frac{1}{3}$.

But with the limit at hand, no such problem will arise if we are just a little careful. We'll use

Lemma. For all $x \neq 0$, we have $1 - \frac{x^2}{6} \leqslant \frac{\sin x}{x} \leqslant 1$.

Proof. Since all terms are even functions of x, it suffices to consider $x > 0$. Define $f(x) = \sin x - x + \frac{x^3}{6}$, so that

$$f'(x) = \cos x - 1 + \tfrac{x^2}{2}, \quad f''(x) = x - \sin x, \quad f'''(x) = 1 - \cos x$$

and $f^{(n)}(0) = 0$ for $n = 0, 1, 2, 3$. Clearly $f'''(x) \geqslant 0$ for all x, so f'' is increasing, so $f'' > 0$ on $(0, \infty)$, so f' is increasing on $(0, \infty)$, so $f' > 0$ on $(0, \infty)$, so $f > 0$ on $(0, \infty)$. Our upper and lower bounds for $\frac{\sin x}{x}$ on $(0, \infty)$ follow from $f'' > 0$ and $f > 0$ respectively. □

Let's define

$$S_N = \sum_{k=1}^{N} \frac{1}{N^2 \sin^2 \frac{(2k-1)\pi}{4N}}.$$

We will apply the Lemma to each term of this sum, where $x = \frac{(2k-1)\pi}{4N} \in (0, \frac{\pi}{2})$. On the interval $(0, \frac{\pi}{2})$, we have $1 - \frac{x^2}{6} > 0$ and so

$$1 \leqslant \frac{x}{\sin x} \leqslant \frac{1}{1 - \frac{x^2}{6}} < 1 + \tfrac{x^2}{6}$$

and

$$1 \leqslant \frac{x^2}{\sin^2 x} < \Big(1 + \tfrac{x^2}{6}\Big)^2 < 1 + x^2.$$

Substituting $x = \frac{(2k-1)\pi}{4N} \in (0, \frac{\pi}{2})$, this yields

$$\frac{16}{(2k-1)^2\pi^2} \leqslant \frac{1}{N^2 \sin^2 \frac{(2k-1)\pi}{4N}} \leqslant \frac{1}{N^2} + \frac{16}{(2k-1)^2\pi^2}$$

for $k = 1, 2, 3, \ldots, N$. Summing over all k yields

$$\frac{16}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2} \leqslant S_N \leqslant \frac{1}{N} + \frac{16}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2}.$$

By the Squeeze Theorem, we have $S_N \to \frac{16}{\pi^2} \sum_{k=1}^{N} \frac{1}{(2k-1)^2}$. From here we obtain Euler's formula $\zeta(2) = \frac{\pi^2}{6}$ as before.

Finally, we consider the formula for general N. Let us denote

$$f(z) = \sum_{k=1}^{2N} \frac{\zeta_{4N}^{2k-1} z}{(z - \zeta_{4N}^{2k-1})^2}.$$

Note that the values ζ_{4N}^{2k-1} (for $k = 1, 2, \ldots, 2N$) are precisely the complex roots of $z^{2N}+1$ and by symmetry we have $f(\zeta_{4N}^2 z) = f(z)$ for all z. The least common denominator of the terms is $(z^{2N}+1)^2$, leading is to consider the polynomial

$$g(z) = (z^{2N}+1)^2 f(z) \in \mathbb{C}[z]$$

of degree at most $4N-1$. Since $g(\zeta_{4N}^2 z) = g(z)$, it follows that $g(z)$ must be a polynomial in z^{2N}. It necessarily must have the form $g(z) = a + bz^{2N}$ for some $a, b \in \mathbb{C}$. Since $g(0) = f(0) = 0$, we have $a = 0$. To determine the coefficient b, we note that

$$bz^{2N} \ = \ (z^{2N}+1)^2 f(z) \ = \sum_{k=1}^{2N} \zeta_{4N}^{2k-1} z \Big(\frac{z^{2N}+1}{z - \zeta_{4N}^{2k-1}} \Big)^2$$

in which we require only the coefficient of z^{2N} on the right side. To simplify notation, we abbreviate $\omega = \zeta_{4N}^{2k-1}$ in the k-th term of the sum. This term has the form

$$\omega z \Big(\frac{z^{2N}+1}{z-\omega} \Big)^2 = \omega z (z^{2N-1} + \omega z^{2N-2} + \omega^2 z^{2N-3} + \omega^3 z^{2N-4} + \cdots + \omega^{2N-2} z + \omega^{2N-1})^2$$

in which the coefficient of z^{2N} is clearly $\omega \cdot 2N \omega^{2N-1} = -2N$. Since there are $2N$ terms in the sum, we have $b = -4N^2$. Evaluating at $z = 1$ gives $4f(1) = g(1) = -4N^2$ so $f(1) = -N^2$. However,

$$f(1) = \sum_{k=1}^{2N} \frac{\zeta_{4N}^{2k-1}}{(1-\zeta_{4N}^{2k-1})^2} = \sum_{k=1}^{2N} \frac{1}{(1-\zeta_{4N}^{2k-1})(\overline{\zeta_{4N}}^{2k-1} - 1)} = \sum_{k=1}^{2N} \frac{1}{|1-\zeta_{4N}^{2k-1}|^2} = 2 \sum_{k=1}^{N} \frac{1}{|1-\zeta_{4N}^{2k-1}|^2}.$$

This proves the claimed identity.